Applied Analysis (APPM 5450): Final

7.30 am – 10.00 am, May 6, 2010. Closed books.

Problem 1: (28p) Four points for each question. No motivation required.

(a) State the axioms for a σ-algebra.

(b) Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$. Which statements are necessarily true:
  (i) If $A^*\,A = I$, then $||A\,x|| = ||x||$ for all $x \in H$.
  (ii) If $||A\,x|| = ||x||$ for all $x \in H$, then $(Ax,\,Ay) = (x,\,y)$ for all $x, y \in H$.
  (iii) If $(Ax,\,Ay) = (x,\,y)$ for all $x, y \in H$, then A is unitary.

(c) Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of Schwartz functions on $\mathbb{R}$ that are all supported in the interval $I = [-1,\,1]$. Suppose further that

$$\lim_{n\to\infty}\left(\sup_{x\in I}|\varphi_n(x) - \varphi(x)|\right) = 0.$$

Which of the following statements are necessarily true:
  (i) $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R})$.
  (ii) $\varphi_n \to \varphi$ in $\mathcal{S}^*(\mathbb{R})$.
  (iii) $\varphi_n \to \varphi$ in norm in $L^p(\mathbb{R})$ for all $p \in [1,\,\infty]$.

(d) Define an operator A on $L^2(\mathbb{R})$ via $[A\,u](x) = \frac{1}{2}\big(u(x) + u(-x)\big)$. (To be rigorous, we could define A on $\mathcal{S}(\mathbb{R})$ and then extend it to $L^2(\mathbb{R})$ via a density argument.) Specify $\sigma(A)$.

(e) Let $p \in [1,\,\infty]$, and define functions $(f_n)_{n=1}^{\infty} \subset L^p(\mathbb{R})$ via $f_n = \frac{1}{\sqrt{n}}\,\chi_{[0,\,n]}$. For which $p \in [1,\,\infty]$ does $(f_n)_{n=1}^{\infty}$ converge weakly?

(f) Define $f \in \mathcal{S}^*(\mathbb{R})$ via $f(x) = \sin(x)$. What is $\hat{f}$?

(g) Let $\mathcal{F} :\, L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform. What do you know about the spectrum of $\mathcal{F}$?

Problem 2: (24p) Set $H = L^2(\mathbb{R})$, and consider for $n = 1,\,2,\,3,\,\dots$ the operator $A_n \in \mathcal{B}(H)$ given by

$$[A_n u](x) = e^{-x^2/2n}\,u(x).$$

Each operator A_n is self-adjoint, and you may use this fact without proving it. Briefly motivate your answers to all questions below ***except part (c):***

(a) (4p) Is A_n compact?

(b) (4p) Is A_n non-negative? Positive? Coercive?

(c) (6p) Specify $\sigma(A_n)$, $\sigma_{\rm p}(A_n)$, $\sigma_{\rm c}(A_n)$, and $\sigma_{\rm r}(A_n)$.

(d) (6p) Does the sequence $(A_n)_{n=1}^{\infty}$ converge in $\mathcal{B}(H)$? If so, specify the limit and the mode of convergence.

(e) (4p) With $\mathcal{F}$ the Fourier transform, describe the operator $\hat{A}_n = \mathcal{F}^*\,A_n\,\mathcal{F} \in \mathcal{B}(H)$. That is, specify the action of $\hat{A}_n$ without referring to $\mathcal{F}$. Does $(\hat{A}_n)_{n=1}^{\infty}$ converge?

Problem 3: (18p) Let p be a real number such that $1 \leq p < \infty$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^p(\mathbb{R})$ that converges pointwise to a function f. In other words,

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for all } x \in \mathbb{R}.$$

Suppose further that all f_n satisfy

$$|f_n(x)| \leq 2|f(x)|, \qquad \text{for all } x \in \mathbb{R}.$$

For each of the three sets of conditions on f given below, specify for which $r \in [1, \infty)$ it is necessarily the case that

$$\lim_{n\to\infty} ||f - f_n||_{L^r(\mathbb{R})} = 0.$$

(a) $|f| \leq \chi_{[-1,\,1]}$.

(b) $f \in L^p(\mathbb{R})$ and $|f(x)| \leq 1$ for all $x \in \mathbb{R}$.

(c) $f \in L^p(\mathbb{R})$.

For each part, three points for a correct answer, and three points for a correct motivation.

Problem 4: (15p) Let $(c_n)_{n=1}^{\infty}$ be a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} n^6\,|c_n|^2 < \infty,$$

and set

$$u(x) = \sum_{n=1}^{\infty} c_n\, e^{i\,n\,x}.$$

For which non-negative integers k is it necessarily the case that $u \in C^k([-\pi,\,\pi])$? Motivate your answer without invoking the Sobolev embedding theorem.

Problem 5: (15p) Define $f \in \mathcal{S}^*(\mathbb{R})$ via $f(x) = |x|/(1 + |x|)$. Calculate the distributional derivatives f' and f''. Please motivate carefully.