Applied Analysis (APPM 5450): Final — Solutions

7.30 am – 10.00 am, May 6, 2010. Closed books.

Problem 1: (28p) Four points for each question. No motivation required.

(a) State the axioms for a σ-algebra.

(b) Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$. Which statements are necessarily true:
  (i) If $A^*A = I$, then $||A\,x|| = ||x||$ for all $x \in H$.
  (ii) If $||A\,x|| = ||x||$ for all $x \in H$, then $(Ax,\, Ay) = (x,\, y)$ for all $x, y \in H$.
  (iii) If $(Ax,\, Ay) = (x,\, y)$ for all $x, y \in H$, then A is unitary.

(c) Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of Schwartz functions on $\mathbb{R}$ that are all supported in the interval $I = [-1,\, 1]$. Suppose further that

$$\lim_{n\to\infty}\left(\sup_{x\in I}|\varphi_n(x) - \varphi(x)|\right) = 0.$$

Which of the following statements are necessarily true:
  (i) $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R})$.
  (ii) $\varphi_n \to \varphi$ in $\mathcal{S}^*(\mathbb{R})$.
  (iii) $\varphi_n \to \varphi$ in norm in $L^p(\mathbb{R})$ for all $p \in [1,\, \infty]$.

(d) Define an operator A on $L^2(\mathbb{R})$ via $[A\,u](x) = \frac{1}{2}\big(u(x) + u(-x)\big)$. (To be rigorous, we could define A on $\mathcal{S}(\mathbb{R})$ and then extend it to $L^2(\mathbb{R})$ via a density argument.) Specify $\sigma(A)$.

(e) Let $p \in [1,\, \infty]$, and define functions $(f_n)_{n=1}^{\infty} \subset L^p(\mathbb{R})$ via $f_n = \frac{1}{\sqrt{n}}\,\chi_{[0,\,n]}$. For which $p \in [1,\, \infty]$ does $(f_n)_{n=1}^{\infty}$ converge weakly?

(f) Define $f \in \mathcal{S}^*(\mathbb{R})$ via $f(x) = \sin(x)$. What is $\hat{f}$?

(g) Let $\mathcal{F} :\, L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform. What is the spectrum of $\mathcal{F}$?

Solution:

(a) See text book.

(b) (i) is TRUE since $||Ax||^2 = (Ax,\, Ax) = (A^*Ax,\, x) = (Ix, x) = ||x||^2$.
(ii) is TRUE due to the polarization identity.
(iii) is FALSE since the condition does not imply that the operator is onto (the right-shift operator on $\ell^2(\mathbb{N})$ provides a counter example).

(c) (i) is FALSE since, for instance, $||\varphi_n - \varphi||_{1,0} = ||\varphi_n' - \varphi'||_{\mathrm{u}}$ need not converge to zero.
(ii) is TRUE.
(iii) is TRUE.

(d) $\sigma(A) = \{0, 1\}$. (Note that A is a projection operator.)

(e) $\boxed{\text{For } p \geq 2.}$ We have $||f_n||_\infty = n^{-1/2}$ so clearly $f_n \to 0$ in L^∞ (in norm, even). For finite p, we have $||f_n||_p = n^{\frac{1}{p}-\frac{1}{2}}$. For $p > 2$, we see that $\lim_{n\to\infty}||f_n|| = 0$, while for $p < 2$, we have $\lim_{n\to\infty}||f_n||_p = \infty$ so (f_n) cannot possibly converge weakly. In the borderline case $p = 2$ we have $||f_n||_2 = 1$, but we can show weak convergence by verifying that $(f_n, g) \to 0$ for all g in a dense subset (such as the compactly supported functions).

(f) $\hat{f} = \frac{\sqrt{2\pi}}{2i}\left(\tau_1\delta - \tau_{-1}\delta\right)$ (so that $\langle\hat{f},\, \varphi\rangle = \frac{\sqrt{2\pi}}{2i}\big(\varphi(-1) - \varphi(1)\big)$). To see this, observe that $\sin(x) = \frac{1}{2i}\big(e^{ix} - e^{-ix}\big)$, that $\mathcal{F}[e^{ikx}\,\varphi] = \tau_k\hat{\varphi}$, and that $\mathcal{F}1 = \sqrt{2\pi}\delta$.

(g) $\sigma(\mathcal{F}) = \sigma_{\mathrm{p}}(\mathcal{F}) = \{1,\, -1,\, i,\, -i\}$. Partial credit is given for the answer that $\sigma(\mathcal{F}) \subseteq \{\lambda \in \mathbb{C} :\, |\lambda| = 1\}$ which you can deduce from the fact that $\mathcal{F}$ is unitary.

Problem 2: (24p) Set $H = L^2(\mathbb{R})$, and consider for $n = 1, 2, 3, \dots$ the operator $A_n \in \mathcal{B}(H)$ given by

$$[A_n u](x) = e^{-x^2/2n}\, u(x).$$

Each operator A_n is self-adjoint, and you may use this fact without proving it. Briefly motivate your answers to all questions below ***except part (c):***

(a) (4p) Is A_n compact?
(b) (4p) Is A_n non-negative? Positive? Coercive?
(c) (6p) Specify $\sigma(A_n)$, $\sigma_{\rm p}(A_n)$, $\sigma_{\rm c}(A_n)$, and $\sigma_{\rm r}(A_n)$.
(d) (6p) Does the sequence $(A_n)_{n=1}^{\infty}$ converge in $\mathcal{B}(H)$? If so, specify the limit and the mode of convergence.
(e) (4p) With $\mathcal{F}$ the Fourier transform, describe the operator $\hat{A}_n = \mathcal{F}^*\, A_n\, \mathcal{F} \in \mathcal{B}(H)$.
That is, specify the action of $\hat{A}_n$ without referring to $\mathcal{F}$. Does $(\hat{A}_n)_{n=1}^{\infty}$ converge?

Solution:

(a) No, A_n is not compact. To prove this, set $\varphi_j = 2^{j/2}\,\chi_{(2^{-j},\,2^{-j+1})}$. Then $(\varphi_j)_{j=1}^{\infty}$ is a bounded sequence, but $(A_n\varphi_j)_{j=1}^{\infty}$ cannot have a convergent subsequence since it is an orthogonal sequence in which the vectors satisfy $||A_n\,\varphi_j|| \geq e^{-1/2}$.

(b) A_n is positive (and hence non-negative). To see this, fix a non-zero vector u. Then pick an R such that $\int_{|x|\leq R} |u(x)|^2\,dx = \epsilon > 0$. Then

$$(A_n u,\, u) = \int_{-\infty}^{\infty} e^{-x^2/2n}|u(x)|^2\,dx \geq \int_{-R}^{R} e^{-x^2/2n}|u(x)|^2\,dx \geq e^{-R^2/2n}\,\varepsilon > 0.$$

To see that A_n is not coercive, set $\psi_j = \chi_{(j,\,j+1)}$. Then $||\psi_j|| = 1$, and $\lim_{j\to\infty} ||A_n\psi_j|| = 0$.

(c) $\sigma(A_n) = \sigma_{\rm c}(A_n) = [0,\,1]$. $\sigma_{\rm p}(A_n) = \sigma_{\rm r}(A_n) = \emptyset$.

(d) (A_n) converges *strongly* to the identity. To prove this, fix any $u \in H$. Then

(1)
$$||A_n u - u||^2 = \int_{-\infty}^{\infty} \left(e^{-x^2/2n} - 1\right)^2 |u(x)|^2\,dx.$$

The integrand in (1) converges pointwise to zero as $n \to \infty$. Moreover, the integrand is dominated by $|u(x)|^2$, and $\int_{\mathbb{R}} |u|^2 < \infty$. Therefore, the LDCT applies, and $\lim_{n\to\infty} ||A_n u - u||^2 = 0$.

To see that (A_n) cannot converge in norm, set $\psi_j = \chi_{(j,\,j+1)}$. Then $||\psi_j|| = 1$, and so $||A_n - I|| \geq ||(A_n - I)\psi_j|| \geq 1 - e^{-j^2/2n}$. Taking the limit as $j \to \infty$, we see $||A_n - I|| \geq 1$.

(e) The key observation is that multiplication by a function in physical space corresponds to convolution in Fourier space. To formalize, set $\varphi_n(x) = e^{-x^2/2n}$, and pick $v \in H$. Then

$$\hat{A}_n\, v = \mathcal{F}^*\,[A_n\,[\mathcal{F}\,v]] = \mathcal{F}^*\,[A_n\,\hat{v}] = \mathcal{F}^*\,[\varphi_n\,\hat{v}] = \sqrt{2\pi}\,\check{\varphi}_n * v.$$

Since $\check{\varphi}_n(t) = \sqrt{n}\,e^{-n\,t^2/2}$, we find

$$[\hat{A}_n\,v](t) = \sqrt{n}\,\sqrt{2\pi}\int_{-\infty}^{\infty} e^{-n\,(t-s)^2/2}\,v(s)\,ds.$$

Finally, observe that since $\mathcal{F}$ is unitary, the convergence properties of $(\hat{A}_n)$ are exactly the same as those of (A_n). In other words, $(\hat{A}_n)$ converges strongly (and not in norm) to $\mathcal{F}^*\,I\,\mathcal{F} = I$.

Problem 3: (18p) Let p be a real number such that $1 \leq p < \infty$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^p(\mathbb{R})$ that converges pointwise to a function f. In other words,

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for all } x \in \mathbb{R}.$$

Suppose further that all f_n satisfy

$$|f_n(x)| \leq 2|f(x)|, \qquad \text{for all } x \in \mathbb{R}.$$

For each of the three sets of conditions on f given below, specify for which $r \in [1, \infty)$ it is necessarily the case that

$$\lim_{n\to\infty} ||f - f_n||_{L^r(\mathbb{R})} = 0.$$

(a) $|f| \leq \chi_{[-1,1]}$.

(b) $f \in L^p(\mathbb{R})$ and $|f(x)| \leq 1$ for all $x \in \mathbb{R}$.

(c) $f \in L^p(\mathbb{R})$.

For each part, three points for a correct answer, and three points for a correct motivation.

Solution: (a) $r \in [1, \infty)$. (b) $r \in [p, \infty)$. (c) $r = p$.

To motivate, we need to prove the claim when it is true, and provide counter-examples when it is not. The basic question we need to resolve is when

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - f_n(x)|^r \, dx = 0. \tag{2}$$

The integrand in (2) converges to zero pointwise, and we want to bring the LDCT to bear. To this end, we construct a dominator h via

$$|f(x) - f_n(x)|^r \leq \big(|f(x)| + |f_n(x)|\big)^r \leq \big(|f(x)| + 2\,|f(x)|\big)^r = 3^r\,|f(x)|^r =: h(x).$$

We will analyze each of the three assumptions to see when $\int h < \infty$.

(a) If $|f| \leq \chi_{[-1,1]}$, then $h \leq 3^r\,\chi_{[-1,1]}$ so $\int h < r^3\,2 < \infty$ and LDCT applies.

(b) Case 1 - $r \geq p$: In this case, $h(x) = 3^r\,|f(x)|^r \leq 3^r\,|f(x)|^p$ since $|f(x)| \leq 1$. Therefore, $\int h \leq r^3 ||f||_p^p < \infty$, and LDCT applies.

Case 2 - $r < p$: In this case, the LDCT does not apply, and we look for a counter-example. Pick a real number α such that $-\frac{1}{r} < \alpha < -\frac{1}{p}$, and set $f(x) = x^{\alpha}\,\chi_{[1,\infty)}$. Then $f \in L^p$. Set $f_n = (1 - 1/n)\,f$. Then $f_n \to f$ pointwise, but $||f - f_n||_r^r = ||(1/n)f||_r^r = \int_1^{\infty} n^{-r}\,x^{\alpha r}\,dx = \infty$.

(c) Case 1 - $r > p$: When $|f|$ is not necessarily bounded, $|f|^r$ is not bounded by $|f|^p$ and the LDCT does not apply. We look for a counter-example. Pick a real number α such that $-\frac{1}{p} < \alpha < -\frac{1}{r}$, and set $f(x) = x^{\alpha}\,\chi_{(0,1)}$. Then $f \in L^p$. Set $f_n = (1 - 1/n)\,f$. Then $f_n \to f$ pointwise, but $||f - f_n||_r^r = ||(1/n)f||_r^r = \int_0^1 n^{-r}\,x^{\alpha r}\,dx = \infty$.

Case 2 - $r = p$: In this case, $\int h = \int 3^p\,|f|^p = 3^p ||f||_p^p < \infty$ so LDCT applies.

Case 3 - $r < p$: In this case, the same counter-example we constructed in part (b) works.

Note: A complete motivation requires counter-examples for the case where the claim does not hold. However, nobody provided them, so only one point was docked for such an omission.

Problem 4: (15p) Let $(c_n)_{n=1}^{\infty}$ be a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} n^6\,|c_n|^2 < \infty,$$

and set

$$u(x) = \sum_{n=1}^{\infty} c_n\, e^{i\,n\,x}.$$

For which non-negative integers k is it necessarily the case that $u \in C^k([-\pi,\,\pi])$? Motivate your answer without invoking the Sobolev embedding theorem.

Solution: $\boxed{\text{For } k = 0,\,1,\,2.}$

Set $u_N = \sum_{n=1}^{N} c_n\, e^{inx}$. Then $u_N \in C^k$ for all k. If we can prove that $(u_N)_{N=1}^{\infty}$ is Cauchy in C^k, then we invoke the fact that C^k is complete to argue that the limit function $u \in C^k$.

Set

$$B = \sum_{n=1}^{\infty} n^6\,|c_n|^2 < \infty,$$

let j be a non-negative integer, and let M and N be integers such that $M < N$. Then for any x we find

$$\begin{aligned}\left|\partial^j\big(u_N(x) - u_M(x)\big)\right| &= \left|\partial^j \sum_{n=M+1}^{N} c_n\, e^{inx}\right| = \left|\sum_{n=M+1}^{N} (in)^j\, c_n\, e^{inx}\right| \leq \sum_{n=M+1}^{N} n^j\,|c_n| \leq \{\text{Cauchy-Schwartz}\}\\ &\leq \left(\sum_{n=M+1}^{N} n^{2j-6}\right)^{1/2} \left(\sum_{n=M+1}^{N} n^6\,|c_n|^2\right)^{1/2} \leq \left(\sum_{n=M+1}^{\infty} n^{2j-6}\right)^{1/2} B = D_{M,j}\,B,\end{aligned}$$

where

$$D_{M,j} = \left(\sum_{n=M+1}^{\infty} n^{2j-6}\right)^{1/2}.$$

It follows that

$$||u_N - u_M||_{C^k} \leq \sum_{j=0}^{k} D_{M,j}\,B.$$

Observe that $\lim_{M\to\infty} D_{M,j} = 0$ when $2j - 6 < -1$. Since j is an integer, this happens when $j = 0,\,1,\,2$.

Note: Most answers to this questions consisted of a demonstration that the sum $\partial^k u = \sum c_n\,(in)^k\, e^{inx}$ converges *in the L^2-norm* when $k \leq 3$. This shows that $u \in H^3$, not that $u \in C^3$. To get to C^3, you need to invoke some type of Sobolev embedding results such as the one used above.

Also note that while the question asked for a *motivation* that did not merely invoke the Sobolev embedding theorem, it can of course be used to arrive at the *correct answer.* The theorem says that $H^m(\mathbb{T}^d) \subset C^k(\mathbb{T}^d)$ when $k < m - d/2$. In our case, we find that $u \in H^3(\mathbb{T}^1)$, so $m = 3$ and $d = 1$. We must have $k < 3 - 1/2$, or, in other words, $k = 0,\,1,\,2$.

Problem 5: (15p) Define $f \in \mathcal{S}^*(\mathbb{R})$ via $f(x) = |x|/(1+|x|)$. Calculate the distributional derivatives f' and f''. Please motivate carefully.

Solution: Observe that $f(x) = \dfrac{1+|x|-1}{1+|x|} = 1 - \dfrac{1}{1+|x|}$.

First we evaluate f'. Fix $\varphi \in \mathcal{S}$. Then

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\underbrace{\int_{-\infty}^{\infty} \varphi'}_{=0} + \int_{-\infty}^{0} \frac{1}{1-x}\,\varphi' + \int_{0}^{\infty} \frac{1}{1+x}\,\varphi'$$

$$= \left[\frac{1}{1-x}\,\varphi\right]_{-\infty}^{0} - \int_{-\infty}^{0} \frac{1}{(1-x)^2}\,\varphi + \left[\frac{1}{1+x}\,\varphi\right]_{0}^{\infty} + \int_{0}^{\infty} \frac{1}{(1+x)^2}\,\varphi$$

$$= \varphi(0) - \int_{-\infty}^{0} \frac{1}{(1-x)^2}\,\varphi - \varphi(0) + \int_{0}^{\infty} \frac{1}{(1+x)^2}\,\varphi = \langle g, \varphi\rangle$$

where $g = f'$ is a regular function given by

$$f'(x) = g(x) = \frac{\mathrm{sign}(x)}{(1+|x|)^2}.$$

(The definition of $g(0)$ is arbitrary.)

Observe that in the calculation above we used that $\lim_{x\to\pm\infty} \varphi(x) = 0$ for any $\varphi \in \mathcal{S}$.

Proceeding to $f'' = g'$, we find

$$\langle f'', \varphi\rangle = \langle g', \varphi\rangle = -\langle g, \varphi'\rangle = \int_{-\infty}^{0} \frac{1}{(1-x)^2}\,\varphi' - \int_{0}^{\infty} \frac{1}{(1+x)^2}\,\varphi'$$

$$= \left[\frac{1}{(1-x)^2}\,\varphi\right]_{-\infty}^{0} - \int_{-\infty}^{0} \frac{2}{(1-x)^3}\,\varphi - \left[\frac{1}{(1+x)^2}\,\varphi\right]_{0}^{\infty} - \int_{0}^{\infty} \frac{2}{(1+x)^3}\,\varphi$$

$$= \varphi(0) - \int_{-\infty}^{0} \frac{2}{(1-x)^3}\,\varphi + \varphi(0) - \int_{0}^{\infty} \frac{2}{(1+x)^3}\,\varphi.$$

We see that

$$f'' = g' = 2\delta + h,$$

where h is a regular function given by

$$h(x) = -\frac{2}{(1+|x|)^3}.$$

Note: Many solutions given involved sign errors, mistaken calculations of the derivative, *etc.* Such errors of course only result in a very minor loss of points, but notice that they are entirely unnecessary. The *signs* are obvious if you simply sketch the graphs of f and f'. Moreover, away from the origin, f is a regular function and its distributional derivatives must coincide with its classical derivatives, which can easily be evaluated.