Homework set 12 — APPM5450, Spring 2010

From the textbook: 12.2, 12.3, 12.5, 12.7.

Problem: Let (X, μ) be a measure space and consider the space $L^\infty(X, \mu)$ consisting of all measurable functions from X to $\mathbb{R}$ such that

$$||f||_\infty = \operatorname*{ess\,sup}_{x \in X} |f(x)| < \infty.$$

Prove that $L^\infty(X, \mu)$ is closed under the norm $|| \cdot ||_\infty$.

Hint: You may want to start as follows:

(1) Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $L^\infty(X, \mu)$.

(2) For each positive integer k, there exists and N_k such that for $m, n \geq N_k$, $||f_n - f_m||_\infty < 1/k$.

(3) For each k, and for each $m, n \geq N_k$, let Ω_{mn}^k denote the set of all $x \in X$ such that $|f_m(x) - f_n(x)| < 1/k$. What can you tell about Ω_{mn}^k in light of (2)?

(4) Set $\Omega^k = \bigcap_{m,n=N_k}^\infty \Omega_{mn}^k$. What do you know about Ω^k in view of your conclusion from (3)?

(5) Set $\Omega = \bigcap_{k=1}^\infty \Omega^k$. What do you know about Ω in view of your conclusion from (4)?

(6) What can you tell about $(f_n(x))_{n=1}^\infty$ for $x \in \Omega$?