Homework set 13 — APPM5450, Spring 2010

From the textbook: 12.4.

Problem 1: Let $(f_n)_{n=1}^{\infty}$ be a sequence of real valued measurable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = x$ for all $x \in \mathbb{R}$. Specify which of the following limits necessarily exist, and give a formula for the limit in the cases where this is possible:

$$\lim_{n\to\infty} \int_1^2 \frac{f_n(x)}{1+f_n(x)^2}\,dx, \tag{1}$$

$$\lim_{n\to\infty} \int_0^1 \frac{\sin(f_n(x))}{f_n(x)}\,dx, \tag{2}$$

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(f_n(x))}{f_n(x)}\,dx, \tag{3}$$

$$\lim_{N\to\infty} \int_0^1 \sum_{n=1}^{N} \frac{|f_n(x)|}{n^2(1+|f_n(x)|)}\,dx, \tag{4}$$

$$\lim_{N\to\infty} \int_0^\infty \sum_{n=1}^{N} \frac{1}{n^2(1+|f_n(x)|^2)}\,dx. \tag{5}$$

Problem 2: Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued measurable functions on $\mathbb{R}$ such that $|f_n(x)| \leq 1$ and $\lim_{n\to\infty} f_n(x) = 1$ for all x. Evaluate

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(\cos x)\, e^{-\frac{1}{2}(x-2\pi n)^2}\,dx.$$

Make sure to justify your calculation.

For further practice on problems involving convergence theorems, please look at older exams for APPM5450, and also at older analysis prelims. You will see that most of these exams involve at least one question on limit theorems, reflecting their importance in the curriculum.