Homework 13

12.4) Give an example of a monotonic decreasing sequence of nonnegative functions converging pointwise to a function f such that the equality in Theorem 12.33 (Monotone convergence) does not hold.

Consider $f_n(x) = \frac{1}{n}$ for all $x \in R$. Then $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \infty$, whereas $\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)dx = 0$.

Problem 1) Let $(f_n)_{n=1}^{\infty}$ be a sequence of real valued measurable functions on R such that $\lim_{n\to\infty} f_n(x) = x$ for all $x \in R$. Specify which of the following limits necessarily exist, and give a formula for the limit in the cases where this is possible:

(1) $\lim_{n\to\infty} \int_1^2 \frac{f_n(x)}{1+f_n(x)^2}dx$

We can bound the integrand: $\left|\frac{f_n(x)}{1+f_n(x)^2}\right| \le \sup_t \frac{|t|}{1+t^2} \le 1$

Then, since $\int_1^2 1dx = 1 < \infty$ dominated convergence applies:

$$\lim_{n\to\infty} \int_1^2 \frac{f_n(x)}{1+f_n(x)^2}dx = \int_1^2 \lim_{n\to\infty} \frac{f_n(x)}{1+f_n(x)^2}dx = \int_1^2 \frac{x}{1+x^2}dx = \left[\frac{\log(1+x^2)}{2}\right]_1^2 = \log\left(\sqrt{\frac{5}{2}}\right)$$

(2) $\lim_{n\to\infty} \int_0^1 \frac{\sin(f_n(x))}{f_n(x)}dx$

We can bound the integrand: $\left|\frac{\sin(f_n(x))}{f_n(x)}\right| \le \left|\frac{\sin(t)}{t}\right| \le 1$

Then, since $\int_0^1 1dx = 1 < \infty$ dominated convergence applies:

$$\lim_{n\to\infty} \int_0^1 \frac{\sin(f_n(x))}{f_n(x)}dx = \int_0^1 \lim_{n\to\infty} \frac{\sin(f_n(x))}{f_n(x)}dx = \int_0^1 \frac{\sin(x)}{x}dx \approx 0.946083$$

(3) $\lim_{n\to\infty}\int_0^\infty \frac{\sin(f_n(x))}{f_n(x)}dx$

We can bound the integrand: $\left|\frac{\sin(f_n(x))}{f_n(x)}\right| \le \left|\frac{\sin(t)}{t}\right| \le 1$

However, since $\int_0^\infty 1dx = \infty$ dominated convergence does not apply.

For this problem we can actually achieve different values for the limit depending on $f_n(x)$.

a) Define $f_n(x) = \begin{cases} x & 0 \le x \le 2\pi n \\ \pi & x > 2\pi n \end{cases}$, then $\lim_{n\to\infty}\int_0^\infty \frac{\sin(f_n(x))}{f_n(x)}dx = \frac{\pi}{2}$

b) Note that $\frac{\sin(f_n(x))}{f_n(x)}$ oscillates about the x-axis with decreasing magnitude. For each n we can construct $f_n(x)$ so that $\frac{\sin(f_n(x))}{f_n(x)}$ is made by adding up 2n sections of area above the x-axis while counting just n sections of area below the x-axis. Then $\lim_{n\to\infty}\int_0^\infty \frac{\sin(f_n(x))}{f_n(x)}dx = \infty$

(4) $\lim_{N\to\infty}\int_0^1 \sum_{n=1}^{N} \frac{|f_n(x)|}{n^2(1+|f_n(x)|)}dx$

Since every term in the sum is non-negative monotonic convergence applies:

$$\lim_{N\to\infty}\int_0^1 \sum_{n=1}^{N} \frac{|f_n(x)|}{n^2(1+|f_n(x)|)}dx = \int_0^1 \sum_{n=1}^{\infty} \frac{|f_n(x)|}{n^2(1+|f_n(x)|)}dx < \infty$$

We know that the limit exists and is finite, but what the actual limit is depends on $(f_n)_{n=1}^\infty$.

(5) $\lim_{N\to\infty}\int_0^\infty \sum_{n=1}^{N} \frac{1}{n^2\left(1+|f_n(x)|^2\right)}dx$

Since every term in the sum is non-negative monotonic convergence applies:

$$\lim_{N\to\infty}\int_0^\infty \sum_{n=1}^{N} \frac{1}{n^2\left(1+|f_n(x)|^2\right)}dx = \int_0^\infty \sum_{n=1}^{\infty} \frac{1}{n^2\left(1+|f_n(x)|^2\right)}dx$$

Once again the limit exists, but now (depending on $(f_n)_{n=1}^\infty$) it might be infinite (the key difference is that the interval is no longer finite). Consider:

a) $f_n(x) = x$ for all n. Then $\int_0^\infty \sum_{n=1}^{\infty} \frac{1}{n^2(1+x^2)}dx = \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty \frac{1}{(1+x^2)}dx = \frac{\pi^2}{6}\frac{\pi}{2} = \frac{\pi^3}{12}$

b) $f_n(x) = \begin{cases} x & 0 \le x \le n \\ 0 & x > n \end{cases}$. Then the integral is infinite.

Problem 2) Let $(f_n)_{n=1}^{\infty}$ be a sequence of real valued measurable functions on R such that $|f_n(x)| \le 1$ and $\lim_{n\to\infty} f_n(x) = 1$ for all $x \in R$. Evaluate the following (justify your calculation):

$$\lim_{n\to\infty} \int_R f_n(\cos x) e^{-\frac{1}{2}(x-2\pi n)^2} dx$$

$$\lim_{n\to\infty} \int_R f_n(\cos x) e^{-\frac{1}{2}(x-2\pi n)^2} dx \overset{y=x-2\pi n}{=} \lim_{n\to\infty} \int_R f_n(\cos(y+2\pi n)) e^{-\frac{1}{2}y^2} dy = \lim_{n\to\infty} \int_R f_n(\cos y) e^{-\frac{1}{2}y^2} dy = (*)$$

Note that the first equality is a substitution and the second uses the periodicity of cosine.

For all y we have $f_n(\cos y) e^{-\frac{1}{2}y^2} \xrightarrow{n\to\infty} e^{-\frac{1}{2}y^2}$ and $\left| f_n(\cos y) e^{-\frac{1}{2}y^2} \right| \le e^{-\frac{1}{2}y^2}$

Then, since $\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy < \infty$, dominated convergence applies:

$$(*) = \lim_{n\to\infty} \int_R f_n(\cos y) e^{-\frac{1}{2}y^2} dy = \int_R \lim_{n\to\infty} f_n(\cos y) e^{-\frac{1}{2}y^2} dy = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = \sqrt{2\pi}$$

Problem 3) The solution to this problem is mostly provided as a hint on the homework page. Below the holes in the solution (given as questions in the hint) are filled in.

(3) What can you tell about Ω_{mn}^k in light of (2)?

You can conclude that $\mu\left(\left(\Omega_{mn}^k\right)^c\right)=0$

(4) What do you know about Ω^k in view of your conclusion from (3)?

$$\mu\left(\left(\Omega^k\right)^c\right)=\mu\left(\bigcup_{m,n=N_k}^{\infty}\left(\Omega_{mn}^k\right)^c\right)\le\sum_{m,n=N_k}^{\infty}\mu\left(\left(\Omega_{mn}^k\right)^c\right)=0$$

(5) What do you know about Ω in view of your conclusion from (4)?

$$\mu\left(\Omega^c\right)=\mu\left(\bigcup_{k=1}^{\infty}\left(\Omega^k\right)^c\right)\le\sum_{k=1}^{\infty}\mu\left(\left(\Omega^k\right)^c\right)=0$$

(6) What can you tell about $\left(f_n(x)\right)_{n=1}^{\infty}$ for $x\in\Omega$?

Because $\left(f_n(x)\right)_{n=1}^{\infty}$ is Cauchy for $x\in\Omega$ it makes sense to define $f(x)=\lim_{n\to\infty}f_n(x)$ in this region.

For $x\in\Omega^c$ we can simply set $f(x)=0$.

Fix $\varepsilon>0$. Pick $k>1/\varepsilon$. Then, for $n\ge N_k$ we have:

$$\left\|f-f_n\right\|_\infty=\underset{x\varepsilon X}{ess\sup}\left|f(x)-f_n(x)\right|\overset{(5)}{=}\underset{x\varepsilon\Omega}{ess\sup}\left|f(x)-f_n(x)\right|\le\sup_{x\varepsilon\Omega}\left|f(x)-f_n(x)\right|=\sup_{x\varepsilon\Omega}\lim_{m\to\infty}\left|f_m(x)-f_n(x)\right|\le$$

$$\le\lim_{m\to\infty}\underbrace{\sup_{x\varepsilon\Omega}\left|f_m(x)-f_n(x)\right|}_{\le\frac{1}{k}\ once\ m\ge N_k}\le\frac{1}{k}<\varepsilon$$

Note that the equality denoted by "(5)" uses $\mu\left(\Omega^c\right)=0$ (proved in (5) above).

Because ε was arbitrary this implies that $\left\|f-f_n\right\|_\infty\to0$