

Homework set 14 — APPM5450, Spring 2010 — Solutions

Problem 12.8: We want to prove that

$$\|f - f_n\|_p^p = \int |f - f_n|^p \rightarrow \infty.$$

We know that $|f - f_n|^p \rightarrow 0$ pointwise, so if we can only justify moving the limit inside the integral, we'll be done.

First note that

$$|f(x)| = \lim_{n \rightarrow \infty} |f_n(x)| \leq |g(x)|.$$

Then we can dominate the integrand as follows:

$$|f - f_n|^p \leq (|f| + |f_n|)^p \leq (|g| + |g|)^p \leq 2^p |g|^p.$$

Since $\int |g|^p < \infty$, we find that the Lebesgue dominated convergence theorem applies, and so

$$\lim_{n \rightarrow \infty} \|f - f_n\|_p^p = \lim_{n \rightarrow \infty} \int |f - f_n|^p = \{\text{LDCT}\} = \int (\lim_{n \rightarrow \infty} |f - f_n|^p) = \int 0 = 0.$$

Problem 12.16: Fix $f \in L^p$ and $\varepsilon > 0$. We want to prove that there exists a $\delta > 0$ such that for $|h| < \delta$, we have $\|f - \tau_h f\|_p < \varepsilon$.

First pick $\varphi \in C_c$ such that $\|f - \varphi\|_p < \varepsilon/3$. Then

$$\begin{aligned} \|f - \tau_h f\|_p &\leq \|f - \varphi\|_p + \|\varphi - \tau_h \varphi\|_p + \|\tau_h \varphi - \tau_h f\|_p \\ &= \|f - \varphi\|_p + \|\varphi - \tau_h \varphi\|_p + \|\varphi - f\|_p < \varepsilon/3 + \|\varphi - \tau_h \varphi\|_p + \varepsilon/3. \end{aligned}$$

Set $R = \sup\{|x| : \varphi(x) \neq 0\}$. Since φ is uniformly continuous, there exists a δ such that if $|x - y| < \delta$, then $|\varphi(x) - \varphi(y)| < \varepsilon/(3\mu(B_{R+1}(0))^{1/p})$. Then, if $h < \min(\delta, 1)$,

$$\|\varphi - \tau_h \varphi\|_p^p = \int_{B_{R+1}(0)} |\varphi(x) - \varphi(x-h)|^p dx < \int_{B_{R+1}(0)} \frac{\varepsilon^p}{3^p \mu(B_{R+1}(0))} dx < \frac{\varepsilon^p}{3^p}.$$

Problem 12.17: For $n = 1, 2, 3, \dots$, set $I_n = (2^{-n}, 2^{-n+1})$, and $f_n = 2^{n/p} \chi_{I_n}$. Then $\|f_n\|_p = 1$ for all n . Suppose $m \neq n$, then

$$\|f_n - f_m\|_\infty = 1,$$

and for $p \in [1, \infty)$ we have

$$\|f_n - f_m\|_p = \left(\int_0^1 (2^n \chi_{I_n} + 2^m \chi_{I_m}) \right)^{1/p} = 2^{1/p}.$$

No subsequence of $(f_n)_{n=1}^\infty$ can be Cauchy, and therefore no subsequence can converge.

Problem 12.18: For $n = 1, 2, 3, \dots$, set $I_n = (2^{-n}, 2^{-n+1})$, and $f_n = 2^n \chi_{I_n}$. Let $(f_{n_j})_{j=1}^\infty$ be a subsequence of $(f_n)_{n=1}^\infty$. Define $g \in L^\infty$ by

$$g = \sum_{j=1}^\infty (-1)^j \chi_{I_{n_j}},$$

and define $\varphi \in (L^1)^*$ via $\varphi(f) = \int f g$. Then $\varphi(f_{n_j}) = (-1)^j$ (verify!) and so (f_{n_j}) cannot converge weakly. Since L^1 is not reflexive, this does not contradict that Banach-Alaoglu theorem.

Problem 12.13: Set $I = [0, 1]$ and let Ω be a dense set in $L^\infty(I)$. For $r \in I$, set $f_r = \chi_{[0, r]}$, and pick $x_r \in \Omega \cap B_{1/3}(f_r)$. Since $\|f_r - f_s\| = 1$ if $s \neq r$, we find that $\|x_r - x_s\| \geq \|f_r - f_s\| - \|f_r - x_r\| - \|f_s - x_s\| \geq 1/3$, so all the x_r 's are distinct. Therefore, Ω must be uncountable, and L^∞ cannot be separable.

To prove that $C(I)$ cannot be dense in $L^\infty(I)$, simply note that if $f = \chi_{[0, 1/2]}$, and $\varphi \in C(I)$, then

$$\|f - \varphi\|_\infty \geq \max(|\varphi(1/2)|, |1 - \varphi(1/2)|) \geq 1/2$$

(verify this!).

An alternative argument for why $C(I)$ cannot be dense in $L^\infty(I)$: If $\varphi_n \in C(I)$, and $\varphi_n \rightarrow f$ in the supnorm, then (φ_n) is a Cauchy sequence with respect to the uniform norm (when applied to continuous functions, the uniform norm and the L^∞ norms are identical). Therefore, there exists a continuous function φ such that $\varphi_n \rightarrow \varphi$ uniformly. Then $f(x) = \varphi(x)$ almost everywhere. But not every equivalence class function in L^∞ has a continuous function in it (for instance $f = \chi_{[0, 1/2]}$).

Problem 12.14: Let p and q be such that $1 \leq p < q \leq \infty$.

First we construct a function $f \in L^p \setminus L^q$. Let α be a non-negative number and set $f(x) = x^{-\alpha} \chi_{[0, 1]}$. Then

$$\|f\|_p^p = \int_0^1 x^{-\alpha p} dx,$$

which is finite if $\alpha p < 1$. Moreover

$$\|f\|_q^q = \int_0^1 x^{-\alpha q} dx$$

which is infinite if $\alpha q > 1$. Consequently, $f \in L^p \setminus L^q$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

To construct a function $f \in L^q \setminus L^p$, set $f = x^{-\alpha} \chi_{[1, \infty)}$. Then

$$\|f\|_p^p = \int_1^\infty x^{-\alpha p} dx$$

which is infinite if $\alpha p < 1$. Moreover

$$\|f\|_q^q = \int_1^\infty x^{-\alpha q} dx$$

which is finite if $\alpha q > 1$. Thus, $f \in L^q \setminus L^p$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

(The arguments above need slight modifications if $q = \infty$, but the idea is the same.)

Consider the function

$$f(x) = \frac{1}{(|x|(1 + \log^2 |x|))^{1/2}}.$$

That $f \in L^2$ is clear, since

$$\begin{aligned} \|f\|_2^2 &= \int_{-\infty}^\infty \frac{1}{|x|(1 + \log^2 |x|)} dx = 2 \int_0^\infty \frac{1}{x(1 + \log^2 x)} dx = \{x = e^t\} \\ &= 2 \int_{-\infty}^\infty \frac{1}{e^t(1 + t^2)} e^t dt = 2\pi. \end{aligned}$$

Moreover, if $p > 2$, then note that there exists a $\delta > 0$ such that

$$x^{(p-2)/2}(1 + \log^2 x)^{p/2} \leq 1$$

when $x \in (0, \delta)$. Then

$$\|f\|_p^p \geq \int_0^\delta \frac{1}{x^{p/2}(1 + \log^2 x)^{p/2}} dx = \int_0^\delta \frac{1}{x} \underbrace{\frac{1}{x^{(p-2)/2}(1 + \log^2 x)^{p/2}}}_{\geq 1} dx = \infty.$$

Analogously, if $p < 2$, then there exists an M such that

$$x^{(p-2)/2}(1 + \log^2 x)^{p/2} \leq 1$$

when $x \geq M$. Then

$$\|f\|_p^p \geq \int_M^\infty \frac{1}{x^{p/2}(1 + \log^2 x)^{p/2}} dx = \int_M^\infty \frac{1}{x} \underbrace{\frac{1}{x^{(p-2)/2}(1 + \log^2 x)^{p/2}}}_{\geq 1} dx = \infty.$$

Problem 12.15: Let $\alpha \in (0, 1)$, and let $m, n \in (1, \infty)$ be such that $1/m + 1/n = 1$ (we will determine suitable values for α, m, n later). Then from Hölder's inequality we obtain

$$(1) \quad \|f\|_r^r = \int |f|^r = \int |f|^{\alpha r} |f|^{(1-\alpha)r} \leq \left(\int |f|^{\alpha m r} \right)^{1/m} \left(\int |r|^{(1-\alpha)n r} \right)^{1/n}.$$

In order to obtain the desired right hand side, we must pick α, m, n so that

$$\begin{aligned} \alpha m r &= p, \\ (1 - \alpha) n r &= q, \\ (1/m) + (1/n) &= 1. \end{aligned}$$

To obtain an equation for α , we eliminate m and n :

$$\frac{(1 - \alpha)r}{q} = \frac{1}{n} = 1 - \frac{1}{m} = 1 - \frac{\alpha r}{p}.$$

Solving for α we obtain

$$\alpha = \frac{pq - pr}{rq - rp} = \frac{1/r - 1/q}{1/p - 1/q}.$$

Equation (1) now takes the form

$$\|f\|_r \leq \left((\|f\|_p^p)^{1/m} (\|f\|_q^q)^{1/n} \right)^{1/r} = \|f\|_p^{p/mr} \|f\|_q^{q/nr}.$$

Finally note that

$$\begin{aligned} \frac{p}{mr} = \alpha &= \frac{1/r - 1/q}{1/p - 1/q}, \\ \frac{q}{nr} = 1 - \alpha &= 1 - \frac{1/r - 1/q}{1/p - 1/q} = \frac{1/p - 1/r}{1/p - 1/q}. \end{aligned}$$