

Homework set 3 — APPM5450, Spring 2010

From the textbook: 8.6. Optional: 8.5.

Problem 1: Let H be a Hilbert space, and let $(\varphi_n)_{n=1}^\infty$ denote an orthonormal basis for H . Given a bounded sequence of complex number $(\lambda_n)_{n=1}^\infty$, define the operator A by setting $Au = \sum_{n=1}^\infty \lambda_n \varphi_n \langle \varphi_n, u \rangle$.

(a) Prove that $\|A\| = \sup_n |\lambda_n|$.

(b) Prove that $A^*u = \sum_{n=1}^\infty \bar{\lambda}_n \varphi_n \langle \varphi_n, u \rangle$. Conclude that A is self-adjoint iff all λ_n 's are real. When is A skew-symmetric?

Problem 2: Consider the Hilbert space $H = L^2([-\pi, \pi])$, and the operator $A \in \mathcal{B}(H)$ defined by $[Au](x) = |x|u(x)$. Prove that A is self-adjoint and positive, but not coercive. Prove that

$$\langle u, v \rangle_A = \langle Au, v \rangle$$

is an inner product on H , but that the topology generated by (the norm generated by) this inner product is *not* equivalent to the topology generated by the L^2 -norm.

Problem 3: Set $H = l^2(\mathbb{Z})$ and let R denote the right-shift operator (so that if $y = Rx$, then $y_n = x_{n-1}$). Construct R^* . Prove that R is unitary. (Recall that the right-shift operator on $l^2(\mathbb{N})$ is *not* unitary!)

Problem 4: Consider the Hilbert space $L^2(\mathbb{T})$. Let k denote a continuous function on \mathbb{T}^2 that takes on complex values. Let A denote the operator $[Au](x) = \int_{\mathbb{T}} k(x, y) u(y) dy$. Prove that $[A^*u](x) = \int_{\mathbb{T}} \bar{k}(y, x) u(y) dy$. Conclude that A is self-adjoint iff $k(x, y) = \overline{k(y, x)} \forall x, y \in \mathbb{T}$.

Problem 5 (optional): Assume that you have proved that in a Hilbert space, $\text{ran}(I - K)$ is closed whenever K has finite rank. Prove from this that $\text{ran}(I - K)$ is closed for all compact operators K . (Then if you feel industrious, feel free to try proving the statement given in the first sentence.)