

**Homework set 7 — APPM5450, Spring 2010**

From the text-book: 9.19, 9.20, 9.22. Optional: 9.21.

**Problem 1:** Consider the Hilbert space  $H = \mathbb{C}^n$ . Let  $A \in \mathcal{B}(H)$ , let  $(e^{(j)})_{j=1}^n$  be the canonical basis, and let  $A$  have the representation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

in the canonical basis. We define the *Hilbert-Schmidt norm* of  $A$  as

$$\|A\|_{\text{HS}} = \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

(a) Let  $(\varphi^{(j)})_{j=1}^n$  be any ON-basis for  $H$ . Show that  $\|A\|_{\text{HS}}^2 = \sum_{j=1}^n \|A\varphi^{(j)}\|^2$ .

(b) Show that  $\|A\| \leq \|A\|_{\text{HS}} \leq \sqrt{n}\|A\|$  for any  $A \in \mathcal{B}(H)$ .

(c) Find  $G, H \in \mathcal{B}(H)$  such that  $\|G\|_{\text{HS}} = \|G\|$  and  $\|H\|_{\text{HS}} = \sqrt{n}\|H\|$ .

**Problem 2:** Let  $H$  be a separable Hilbert space, and let  $A \in \mathcal{B}(H)$ . Suppose that  $H$  has an ON-basis  $(\varphi^{(j)})_{j=1}^{\infty}$  such that

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 < \infty.$$

Prove that if  $(\psi^{(j)})_{j=1}^{\infty}$  is any other ON-basis, then

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{\infty} \|A\psi^{(j)}\|^2.$$

**Problem 3:** Consider the linear space  $L = \mathbb{R}^2$ . Define for  $x = (x_1, x_2) \in L$  the seminorms

$$p_1(x) = |x_1|, \quad p_2(x) = |x_2|.$$

Construct for  $x \in L$ ,  $j \in \{1, 2\}$ , and  $\varepsilon \in (0, \infty)$ , the sets

$$\mathcal{B}_{x,j,\varepsilon} = \{y \in L : p_j(x - y) < \varepsilon\}.$$

Describe these sets geometrically. What is the topology generated by the collection of semi-norms  $\{p_1\}$ ? Is it Hausdorff? What is the topology generated by the collection of semi-norms  $\{p_1, p_2\}$ ? Is it Hausdorff?