Applied Analysis (APPM 5450): Midterm 3 — Solutions
8.30am – 9.50pm, April 19, 2010. Closed books.

Problem 1: (15 points) Let $g, h \in L^2(\mathbb{R})$ and set $f = g * h$. Prove that $||f||_{\mathrm{u}} \leq ||g||_{L^2}\,||h||_{L^2}$ (where $||f||_{\mathrm{u}} = \sup_x |f(x)|$). Is it necessarily the case that $f \in C_0(\mathbb{R})$? Motivate your answer briefly.

Solution:

We have

$$|f(x)| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixt}\hat{f}(t)\,dt\right| \leq \int_{-\infty}^{\infty}\frac{|\hat{f}(t)|}{\sqrt{2\pi}}\,dt.$$

Now $\hat{f}(t) = \sqrt{2\pi}\,\hat{g}(t)\,\hat{h}(t)$ so

$$|f(x)| \leq \int_{-\infty}^{\infty}|\hat{g}(t)\,\hat{h}(t)|\,dt \leq \{\text{Cauchy-Schwartz}\} \leq ||\hat{g}||_{L^2}\,||\hat{h}||_{L^2} = ||g||_{L^2}\,||h||_{L^2},$$

where in the last equality we used that the Fourier transform preserves the L^2-norm.

The Riemann-Lebesgue lemma asserts that if $\hat{f} \in L^1$, then $f \in C_0$. The calculation above tells us that $||\hat{f}||_{L^1} \leq ||g||_{L^2}\,||h||_{L^2}$, so yes, $f \in C_0$.

Note: The inequality can easily be proven in physical space. Simply observe that

$$|f(x)| \leq \int_{-\infty}^{\infty}|g(x-y)|\,|h(y)|\,dy \leq \{\text{Cauchy-Schwartz}\}$$
$$\leq \left(\int_{-\infty}^{\infty}|g(x-y)|^2dy\right)^{1/2}\left(\int_{-\infty}^{\infty}|h(y)|^2dy\right)^{1/2} = ||g||_{L^2}\,||h||_{L^2}.$$

However, some Riemann-Lebesgue-type argument is required in order to say that $f \in C_0$.

Problem 2: (26 points) In this problem, $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is the Schwartz space over the real line, a is a non-zero real number, and $\mathcal{F}$ is the Fourier transform.

(a) [6p] Define the operator $D_a : \mathcal{S} \to \mathcal{S}$ via $[D_a\varphi](x) = \varphi(a\,x)$. Show that for some $b, c \in \mathbb{R}$

$$\mathcal{F}\,D_a\,\varphi = b\,D_c\,\mathcal{F}\,\varphi. \tag{1}$$

(b) [6p] State the appropriate definition of the operator $D_a : \mathcal{S}^* \to \mathcal{S}^*$, and derive for $T \in \mathcal{S}^*$ a formula for $\mathcal{F}D_aT$ analogous to (1). Be careful in motivating your work!

(c) [6p] Fix a function $h \in C_{\rm b}(\mathbb{R})$ (*i.e.* h is bounded and continuous), and set $f_n = D_{1/n}h$ for $n = 1, 2, 3, \dots$. Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges in $\mathcal{S}^*$ and give the limit.

(d) [6p] With f_n as in (c), set $\hat{f}_n = \mathcal{F}f_n$. Does the sequence $(\hat{f}_n)_{n=1}^{\infty}$ converge in $\mathcal{S}^*$? If so, to what?

(e*) [2p] Give an example of a distribution $h \in \mathcal{S}^*$ such that $(D_{1/n}h)_{n=1}^{\infty}$ does not converge in $\mathcal{S}^*$.

Solution:

(a) With the change of variables $y = ax$ we find

$$[\mathcal{F}\,D_a\,\varphi](t) = \frac{1}{\sqrt{2\pi}}\int e^{-ixt}\varphi(ax)\,dx = \frac{1}{\sqrt{2\pi}}\int e^{-iyt/a}\varphi(y)\,dy/a = \frac{1}{a}\hat{\varphi}(t/a) = \left[\frac{1}{a}\,D_{1/a}\,\mathcal{F}\,\varphi\right](t).$$

(b) To heuristically figure out the formula, we first consider the case where T is given by a regular function f (say $f \in C_{\rm c}(\mathbb{R})$ or some such). Then

$$\langle D_af,\,\varphi\rangle = \int f(ax)\,\varphi(x)\,dx = \{y = ax\} = \int f(y)\,\varphi(y/a)\,dy/a = \langle f,\,(1/a)D_{1/a}\varphi\rangle.$$

This inspires the formal definition:

For $T \in \mathcal{S}^*$ and $a \in \mathbb{R}\backslash\{0\}$, define D_aT via $\langle D_aT,\,\varphi\rangle = \langle T,\,(1/a)\,D_{1/a}\,\varphi\rangle$.

We now get

$$\langle \mathcal{F}\,D_aT,\,\varphi\rangle \overset{(1)}{=} \langle D_aT,\,\hat{\varphi}\rangle \overset{(2)}{=} \langle T,\,(1/a)\,D_{1/a}\,\hat{\varphi}\rangle \overset{(3)}{=} \langle T,\,\mathcal{F}\,D_a\,\varphi\rangle \overset{(1)}{=} \langle \mathcal{F}\,T,\,D_a\,\varphi\rangle \overset{(2)}{=} \langle (1/a)\,D_{1/a}\,\mathcal{F}\,T,\,\varphi\rangle.$$

The relations (1) use the definition of $\mathcal{F}$ for a distribution. The relations (2) use the definition of D_a for a distribution. The relation (3) uses the result proven in (a).

(c) First we perform a heuristic calculation to see what the limit should be

$$\lim_{n\to\infty}\langle f_n,\,\varphi\rangle = \lim_{n\to\infty}\int h(x/n)\,\varphi(x)\,dx \overset{\text{maybe?}}{=} \int \lim_{n\to\infty} h(x/n)\,\varphi(x)\,dx \int h(0)\,\varphi(x)\,dx = \langle h(0),\,\varphi\rangle.$$

In other words, if (f_n) is to converge, it appears to converge to the constant function $h(0)$. Now let us prove this rigorously.

Fix $\varphi \in \mathcal{S}$. Set $M = ||h||_{\rm u}$ and $L = ||\varphi||_{L^1}$ (both M and L are finite). Fix an arbitrary $\varepsilon > 0$. We need to find an N such that

$$n \geq N \quad \Rightarrow \quad |\langle f_n,\,\varphi\rangle - \langle h(0),\,\varphi\rangle| = \left|\int (h(x/n) - h(0))\,\varphi(x)\,dx\right| < \varepsilon. \tag{2}$$

We first split the integral into a two parts:

$$(3) \qquad |\langle f_n,\, \varphi\rangle - \langle h(0),\, \varphi\rangle| \le \underbrace{\int_{|x|>R} |h(x/n) - h(0)|\,|\varphi(x)|\,dx}_{:=I_1} + \underbrace{\int_{|x|\le R} |h(x/n) - h(0)|\,|\varphi(x)|\,dx}_{:=I_2}.$$

Pick the split point R such that $\int_{|x|\ge R} |\varphi(x)|\,dx < \varepsilon/(4M)$. Then

$$(4) \qquad I_1 = \int_{|x|>R} |h(x/n) - h(0)|\,|\varphi(x)|\,dx \le \int_{|x|>R} 2\,M\,|\varphi(x)|\,dx < 2\,M\,\frac{\varepsilon}{4\,M} = \varepsilon/2.$$

Since h is continuous, there is a $\delta > 0$ such that $|h(y) - h(0)| < \varepsilon/(2L)$ whenever $|y| \le \delta$. Pick N such that $N > R/\delta$. Then for $n \ge N$, we have

$$(5) \qquad I_2 = \int_{|x|\le R} |h(x/n) - h(0)|\,|\varphi(x)|\,dx < \int_{|x|\le R} \frac{\varepsilon}{2L}\,|\varphi(x)|\,dx \le \varepsilon/2.$$

Combining (3), (4), and (5), we see that (2) must hold.

(d) Since $\mathcal{F}$ is a continuous map from $\mathcal{S}^*$ to $\mathcal{S}^*$, our proof in (c) that $f_n \to f$ immediately implies that $\hat{f}_n \to \hat{f}$, so all that remains is to determine $\hat{f}$. We find that

$$\langle \hat{f},\, \varphi\rangle = \langle f,\, \hat{\varphi}\rangle = \langle h(0),\, \hat{\varphi}\rangle = h(0) \int \hat{\varphi}(t)\,dt = \sqrt{2\pi}\,h(0)\,\varphi(0) = \langle \sqrt{2\pi}\,h(0)\,\delta,\, \varphi\rangle,$$

so $\hat{f}_n \to \sqrt{2\pi}\,h(0)\,\delta$.

(e) We saw in (c) that the continuity of h was key to the convergence of (f_n). In consequence, we try an h that is very much not continuous at the origin, the delta function. With $h = \delta$, we find

$$\langle f_n,\, \varphi\rangle = \langle D_{1/n}\delta,\, \varphi\rangle = \langle \delta,\, n\,D_n\,\varphi\rangle = n\,\varphi(0).$$

We see that the sequence $(\langle f_n,\, \varphi\rangle)_{n=1}^{\infty}$ does not converge (unless $\varphi(0)$ happens to be zero).

Problem 3: (25 points)

(a) [5p] For d a positive integer, and s a real number, define the Sobolev space $H^s(\mathbb{R}^d)$.

(b) [5p] For which s, if any, is it necessarily the case that all functions in $H^s(\mathbb{R}^d)$ are continuous?

(c) [10p] Let $f \in L^2(\mathbb{R})$. Show that the equation $-u'' + u = f$ has a unique solution $u \in H^2(\mathbb{R})$.

(d*) [5p] Give an example of a function $f \in L^2(\mathbb{R}^2)$ such that the equation

$$-\frac{\partial^2 u}{\partial x_1^2} + u = f,$$

does not have a solution in $H^2(\mathbb{R}^2)$.

Solution:

(a) $H^s(\mathbb{R}^d)$ is the set of "all" functions f such that $\int_{\mathbb{R}^d} (1 + |t|^2)^s \, |\hat{f}(t)| \, ds < \infty$.

(To be precise, $H^s(\mathbb{R}^d)$ is the Fourier image of the set of all measurable complex-valued functions f on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} (1 + |t|^2)^s \, |f(t)| \, ds < \infty$.)

(b) By Sobolev's inequality: For $s > d/2$.

(c) In the Fourier domain, the equation reads

$$(t^2 + 1)\,\hat{u}(t) = \hat{f}(t).$$

We immediately see that the function

$$u(x) = \left[\mathcal{F}^* \frac{\hat{f}(t)}{1+t^2}\right](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixt} \frac{\hat{f}(t)}{1+t^2} \, dt$$

solves the equation. We find that

$$||u||_{H^2}^2 = \int_{-\infty}^{\infty} (1+t^2)^2 \, |\hat{u}(t)|^2 \, dt = \int_{-\infty}^{\infty} (1+t^2)^2 \, \frac{|\hat{f}(t)|^2}{(1+t^2)^2} \, dt = \int_{-\infty}^{\infty} |\hat{f}(t)|^2 \, dt = ||f||_{L^2}^2$$

so $u \in H^2$. To show uniqueness, simply note that if u and v both solve the equation, then

$$(t^2 + 1)\,\hat{u} = \hat{f} \quad \text{and} \quad (t^2 + 1)\,\hat{v} = \hat{f} \quad \Rightarrow \quad (t^2 + 1)\,(\hat{u} - \hat{v}) = 0 \quad \Rightarrow \quad \hat{u} = \hat{v}, \quad \Rightarrow \quad u = v.$$

(d) The problem here is that while the equation is smoothing in the x_1-direction, it does precisely nothing in the x_2-direction. There are many ways of constructing counter-examples, but we could for instance set

$$\hat{f}(t_1, t_2) = \begin{cases} \frac{1}{(1+t_2)^{1/2}} & |t_1| \leq 1, \\ 0 & |t_1| > 1. \end{cases}$$

Then

$$||f||_{L^2}^2 = ||\hat{f}||_{L^2}^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{f}(t)|^2 \, dt_1 \, dt_2 = \int_{-\infty}^{\infty} \int_{-1}^{1} \frac{1}{1+t_2^2} \, dt_1 \, dt_2 = 2 \int_{-\infty}^{\infty} \frac{1}{1+t_2^2} \, dt_2 = 2\pi$$

so $f \in L^2(\mathbb{R}^2)$. The solution u of the given equation satisfies

$$\hat{u}(t_1, t_2) = \begin{cases} \frac{1}{1+t_1^2} \frac{1}{(1+t_2)^{1/2}} & |t_1| \leq 1, \\ 0 & |t_1| > 1. \end{cases}$$

We see that $u \notin H^2(\mathbb{R}^2)$ since

$$||u||_{H^2}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|t|^2)^2\,|\hat{u}(t)|^2\,dt_1\,dt_2 = \int_{-\infty}^{\infty}\int_{-1}^{1}\frac{(1+t_1^2+t_2^2)^2}{(1+t_1^2)^2\,(1+t_2^2)}\,dt_1\,dt_2$$
$$\geq \int_{-\infty}^{\infty}\int_{-1}^{1}\frac{(1+0+t_2^2)^2}{(1+1)^2\,(1+t_2^2)}\,dt_1\,dt_2 = \frac{1}{2}\int_{-\infty}^{\infty}(1+t_2^2)\,dt_2 = \infty.$$

Problem 5: (12 points) Let $\mathbb{N}$ denote the set of positive integers, and let $\mathcal{A}$ denote the collection of all subsets of $\mathbb{N}$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of real numbers, and define a function

$$\mu : \mathcal{A} \to \mathbb{R} : \Omega \mapsto \sum_{n\in\Omega}\alpha_n.$$

Under what conditions on the numbers (α_n) is μ a measure? Is it ever a finite measure? Is it ever a σ-finite measure? No motivation required.

Solution:

μ is a measure if and only if all α_n are non-negative. [1]

μ is a finite if and only if $\sum_{n=1}^{\infty}\alpha_n$ is finite.

μ is always σ-finite since $\mathbb{N} = \bigcup_{n=1}^{\infty}\{n\}$ and $\mu(\{n\}) = \alpha_n < \infty$.

[1] So called *signed measures* (and even complex valued measures) do exist but are not covered in this class.