The Sobolev Spaces $H^k(\mathbb{T})$

j'th derivative of f.

Set $C^k(\mathbb{T})$ = Space of functions on $\mathbb{T}$ s.t. $f^{(j)} \in C(\mathbb{T})$ for $j = 0, 1, 2, \dots, k$

$$\|f\|_{C^k} = \sum_{j=0}^{k} \|f^{(j)}\|_u$$

Suppose that $f \in C^1(\mathbb{T})$, and that $f = \sum \alpha_n e_n$ $\quad f' = \sum \beta_n e_n$.

Then $\beta_n = \int \frac{e^{-inx}}{\sqrt{2\pi}} f'(x)\,dx = \left[\frac{e^{-inx}}{\sqrt{2\pi}} f(x)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{(-in)e^{-inx}}{\sqrt{2\pi}} f(x)\,dx = in\alpha_n$

So if f has Fourier series (α_n), then f' has F-series $(in\alpha_n)$.

If $f = \sum \alpha_n e_n$ is a function in $L^2(\mathbb{T}) \setminus C^1(\mathbb{T})$ for which $\sum n^2 |\alpha_n|^2 < \infty$, then we define the <u>weak derivative</u> of f by

$$f' = \sum in\alpha_n e_n$$

We set $H^1(\mathbb{T})$ = All functions $f = \sum \alpha_n e_n \in L^2$ for which $\sum n^2 |\alpha_n|^2 < \infty$

and $\langle f, g \rangle_{H^1} = \sum_n (1 + |n|^2) \bar{\alpha}_n \beta_n$, where $f = \sum \alpha_n e_n$, $g = \sum \beta_n e_n$.

By Parseval: $\langle f, g \rangle_{H^1} = \sum \bar{\alpha}_n \beta_n + \sum \overline{in\alpha_n}\, in\beta_n =$

$$= \int \left[\overline{f(x)} g(x) + \overline{f'(x)}\, g'(x)\right] dx$$

$f = \sum \alpha_n e_n \quad f' = \sum in\alpha_n e_n \quad \bar{f}' = \sum -in\bar{\alpha}_n e_{-n} = \sum in\bar{\alpha}_{-n} e_n$

Moreover, $\int f'(x) g(x)\,dx = \langle \bar{f}', g \rangle = \sum (\overline{in\bar{\alpha}_{-n}}) \beta_n = -\sum in\alpha_{-n}\beta_n = -\sum \alpha_{-n}(in\beta_n) = -\langle \bar{f}, g' \rangle = -\int f(x) g'(x)\,dx$

~~$\int f'(x) g(x) dx = \langle \bar{f}', g \rangle = \sum \overline{in\alpha_n}\, \beta_n = -\sum \bar{\alpha}_n (in\beta_n) = \langle \bar{f}, g' \rangle = \int f(x) g'(x) dx$~~

This is integration by parts!

(The boundary terms vanish thanks to periodicity.)

<u>Aside:</u> Weak derivatives can be defined without Fourier-methods.

Suppose that $f \in L^2(\mathbb{I})$ is a function s.t.

$$\left| \int_{\mathbb{I}} \bar{f}(x)\varphi'(x)\,dx \right| \leq M \|\varphi\|_2 \qquad \forall \varphi \in C^\infty(\mathbb{I}).$$

Then the map F ($F(\varphi) = -\int \bar{f}\varphi'$) is a bdd linear functional defined on $C^\infty(\mathbb{I})$, which is a dense subset of $L^2(\mathbb{I})$ (since $\mathcal{P} \subset C^\infty(\mathbb{I})$).

It can be extended to a map $\bar{F} \in L^2(\mathbb{I})^*$.

By the Riesz repr^n thm, $\exists!\, h \in L^2(\mathbb{I})$ s.t.

$\bar{F}(g) = \langle h, g \rangle \quad \forall\, g \in L^2(\mathbb{I})$.

This g is the weak derivative of f.

Note that $\quad \langle h, \varphi \rangle = \bar{F}(\varphi) = -\langle f, \varphi' \rangle \quad \forall \varphi \in C^\infty(\mathbb{I})$

$\boxed{\text{End of aside}}$

More generally, define for $k \geq 0$:

$H^k(\mathbb{I})$ = The space of all functions $f = \sum \alpha_k e_k$ for which $\sum (1+|n|^{2k})|\alpha_n|^2 < \infty$

For $f, g \in H^k$, set $\langle f, g \rangle_{H^k} = \sum (1+|n|^{2k}) \overline{\alpha_n} \beta_n$ ~~$\Rightarrow \|f\|_{H^k} = (\sum$~~

~~Lemma Suppose that $f = \sum \alpha_n e_n \in H^k(\mathbb{I})$ for some $k > 1/2$.~~

~~Set $f_N = \sum_{n=-N}^{N} \alpha_n e_n$. Then $\exists\, C_k < \infty$ s.t.~~

~~$\|f - f$~~

Lemma Suppose that $k > 1/2$. Then $\exists\, C_k < \infty$ w/ the following prop^s:
For every $f \in H^k(\mathbb{T})$, we have

$$\|f - f_N\|_u \leq \frac{C_k}{N^{k-1/2}} \|f\|_{H^k}$$

where $f_N = \sum_{n=-N}^{N} \alpha_n e_n$

In other words, the lemma asserts that if $f \in H^k$ for $k > 1/2$, then the Fourier series converges uniformly to f.
Since each f_N is continuous, this proves that $f \in C(\mathbb{T})$.
(and so ~~$C(\mathbb{T}$~~ $H^k(\mathbb{T}) \subset C(\mathbb{T})$). ~~This is a special case of:~~

~~Thm~~
~~Iso~~

Proof: First we prove that $(f_N)_{N=1}^{\infty}$ is a Cauchy seq in $C(\mathbb{T})$.
If $N < M$, then

$$\|f_N - f_M\|_u = \sup_x \left| \sum_{N<|j|\leq M} \alpha_j \frac{e^{ijx}}{\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2\pi}} \sum_{|j|>N} |\alpha_j| =$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{|j|>N} j^k |\alpha_j| \frac{1}{j^k} \leq \frac{1}{\sqrt{2\pi}} \underbrace{\left(\sum j^{2k}|\alpha_k|^{2k}\right)^{1/2}}_{\leq \|f\|_{H^k}} \left(\sum \frac{1}{j^{2k}}\right)^{1/2} \leq$$

$$\leq \frac{1}{\sqrt{2\pi}} \|f\|_{H^k} \left(\int_N^\infty \frac{1}{t^{2k}}\,dt\right)^{1/2} = \frac{1}{\sqrt{2\pi}} \|f\|_{H^k} \left(\left[-\frac{1}{(2k-1)t^{2k-1}}\right]_N^\infty\right)^{1/2} =$$

$$= \frac{1}{\sqrt{2\pi}} \|f\|_{H^k} \frac{1}{\sqrt{2k-1}} \frac{1}{N^{k-1/2}}$$

Set $C_k = \frac{1}{\sqrt{2\pi}\sqrt{2k-1}}$.
Since $C(\mathbb{T})$ is complete $\exists!\, g \in C(\mathbb{T})$ s.t. $f_N \to g$ uniformly.
But then $f_N \to g$ in L^2 as well, so we must have $g = f$.
Finally note that

$$\|f - f_N\|_u = \lim_{M\to\infty} \|f_M - f_N\|_u \leq \frac{C_k}{N^{k-1/2}} \|f\|_{H^k}$$

We proved that $H^k(\mathbb{T}) \subseteq C(\mathbb{T})$ when $k > 1/2$.

More generally we have

Thm (Sobolev embedding) Suppose that d is a positive integer, and that $k > d/2$. Then $H^k(\mathbb{T}^d) \subseteq C(\mathbb{T}^d)$.

Moreover, the map

$$E : H^k(\mathbb{T}^d) \to C(\mathbb{T}^d) : f \mapsto f$$

is compact.

SEC 7.3 - FOURIER METHODS FOR SOLVING PDES

Let us start with a brief review of techniques for solving linear systems of ODEs.

Let A be a symmetric $N \times N$ matrix and consider the eq^n

$$\text{(ODE)} \begin{cases} Au(t) = \frac{d}{dt} u(t) \\ u(0) = f \end{cases}$$

for the vector valued function $u = u(t) \in \mathbb{R}^N$.

Spectral thm: There is an ON-basis $(\varphi_n)_{n=1}^N$ s.t. $A\varphi_n = \lambda_n \varphi_n$ for some $\lambda_n \in \mathbb{R}$.

Set $u(t) = \sum_{n=1}^N \alpha_n(t) \varphi_n$ for some functions $\alpha_n(t)$ to be determined.

We find

$$\left.\begin{aligned} Au &= \sum_{n=1}^N \alpha_n(t) A\varphi_n = \sum_{n=1}^N \alpha_n(t) \lambda_n \varphi_n \\ \frac{d}{dt} u &= \sum_{n=1}^N \alpha_n'(t) \varphi_n \end{aligned}\right\} \Rightarrow \alpha_n'(t) = \lambda_n \alpha_n(t) \text{ for } n = 1, 2, \dots, N.$$

We see that $\alpha_n(t) = c_n e^{\lambda_n t}$ for some numbers (c_n).

To determine c_n, use the initial condition:

$$u(0) = \sum_{n=1}^N c_n e^{\lambda_n t} \varphi_n \Big|_{t=0} = \sum_{n=1}^N c_n \varphi_n = f$$

so $\quad c_n = \langle \varphi_n, f \rangle \quad$ (since $(\varphi_n)_{n=1}^N$ is an ON-basis)

We find that the unique sol^n is

$$u(t) = \underbrace{\sum_{n=1}^{N} \langle \varphi_n, f \rangle e^{\lambda_n t} \varphi_n}_{= e^{At} f}$$

$= e^{At} f$ by def^n of the matrix exponential.

Now let us try to emulate the same technique for the "heat equation". First we do it heuristically!

Set $I = [0, \pi]$ and let $u = u(t,x)$ be an unknown function. The heat eq^n reads

$$\frac{d^2}{dx^2} u = \frac{d}{dt} u \quad \text{for } t>0,\ x \in I$$

$$\left.\begin{array}{l} u(t,0) = 0 \\ u(t,\pi) = 0 \end{array}\right\} \text{homogeneous boundary conditions}$$

$$u(0,x) = f(x) \quad \leftarrow \text{initial condition}$$

Set $A = \frac{d^2}{dx^2}$ and $\varphi_n(x) = \sin(nx)$.

Observe that $A\varphi_n = -n^2\varphi_n$. Set $\lambda_n = -n^2$ ← evals of $\frac{d^2}{dx^2}$!

Now make the Ansatz $u(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin(nx)$.

$$\left.\begin{array}{l} u_{xx} = \sum_{n=1}^{\infty} -n^2\alpha_n(t)\sin(nx) \\ u_t = \sum_{n=1}^{\infty} \alpha_n'(t)\sin(nx) \end{array}\right] \Rightarrow \alpha_n(t) = c_n e^{-n^2 t}$$

Initial cond^n: $\sum_{n=1}^{\infty} c_n \sin(nx) = f(x) \Rightarrow c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

So $$u(x,t) = \underbrace{\sum_{n=1}^{\infty} \frac{\langle \varphi_n, f\rangle}{\|\varphi_n\|^2} e^{-n^2 t}\sin(nx)}_{= e^{At} f \text{ by def}^n!}$$

Formalize the ~~eqn~~ sol^n of heat eq^n using Fourier methods.

For $t \geq 0$, let $u(t)$ be a function on $\mathbb{T}$ ← the torus

$$u_{xx} = u_t$$
$$u(t=0) = f$$

Assume for now that $f \in \mathbb{P}$ ← the trigonometric polynomials.

If $f = \sum_{n=-N}^{N} \alpha_n e^{inx}$, set $u(x,t) = \sum_{n=-N}^{N} e^{-n^2 t} \underbrace{(\varphi_n, f)}_{=\alpha_n} \varphi_n$

where $\varphi_n(x) = e^{inx}$. This is a classical sol^n.

Define $T(t): \mathbb{P} \to L^2$ by $f \mapsto \sum_{n=-N}^{N} \underbrace{(\varphi_n, f)}_{=\alpha_n} e^{-n^2 t} \varphi_n$.

$$\|T(t) f\|_{L^2}^2 = \sum_{-N}^{N} |\alpha_n e^{-n^2 t}|^2 \leq \sum_{-N}^{N} |\alpha_n|^2 = \|f\|^2 \Rightarrow \|T(t)\| \leq 1.$$

Since $T(t)$ is cont & $\mathbb{P}$ is dense, $T(t)$ can be extended to all of $L^2(\mathbb{T})$.

Properties of $T(t)$:

* $T(0) = I$
* $T(t)T(s) = T(t+s)$ for $s, t \geq 0$
* $T(t) \to I$ strongly as $t \to 0$.

Note that $T(t)$ does not converge in norm at $t \to 0$.

We call $(T(t))_{t \geq 0}$ a strongly continuous semigroup.

For $t > 0$, $T(t) f$ is a classical sol^n of (PDE).

To see this, we prove that $T(t) f \in C^\infty$.

Fix n, then $T(t) f \in H^n(\mathbb{T})$ since

$$\|T(t) f\|_{H^n}^2 = \sum_{n=1}^{\infty} n^2 |\alpha_n|^2 e^{-2n^2 t} \leq C \sum_{n=1}^{\infty} |\alpha_n|^2 = C \|f\|_{L^2}^2$$

Thus $T(t) f \in C^{n-1}(\mathbb{T})$ for all n.

We have constructed <u>one</u> sol^n of PDE.

It remains to prove uniqueness.

$$u_{xx} = u_t \Rightarrow \int_0^{\pi} u\, u_{xx} = \int_0^{\pi} u u_t \Rightarrow -\int_0^{\pi} u_x^2\, dx = \frac{1}{2}\frac{d}{dt}\int_0^{\pi} u^2 = \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2$$

Thus $\|u\|_{L^2}$ is non-increasing as $t \to \infty$. $\Rightarrow \|u(t)\| \le \|f\|$.

Now assume u & v both solve PDE and set $w = u - v$.

Then

$$\left.\begin{array}{l} w_{xx} = w_t \\ w(t=0) = 0 \end{array}\right\} \Rightarrow \|w\|_{L^2} \le \|0\| = 0.$$

PROJECTIONS ON LINEAR SPACES (NO TOPOLOGY)

Def^n Let X be a linear space. An operator $P: X \to X$ is a proj^n if $P^2 = P$.

Lemma 1 Let P be a proj^n on a linear space X.
Set $M = \text{ran}\, P$, $N = \ker P$. Then

(i) $M = \{x \in X : x = Px\}$

(ii) $X = \text{span}(M, N)$

(iii) $M \cap N = \{0\}$

$\Bigg\} \Rightarrow X = \text{ran}\, P \oplus \ker P$

Proof (i) Set $A = \{x : x = Px\}$.
Obviously, $A \subseteq M$.
Conversely, assume $x \in M$, then $\exists y$ s.t. $x = Py \Rightarrow Px = P^2 y = Py = x$

(ii) Given x, set $y = Px$ & $z = x - y$.
Then $x = y + z$, $y \in M$ & $Pz = Px - Py = Px - P^2 x = 0$ so $z \in N$.

(iii) Suppose $x \in M \cap N$. Then $\underset{\text{since } x \in M}{x = Px} \underset{\text{since } x \in N}{= 0}$.

Lemma 2 Suppose that X is a linear space with subspaces M & N s.t. $X = M \oplus N$.
Then $\exists$ a proj^n P s.t. $\text{ran}\, P = M$, $\ker P = N$.

PROJECTIONS ON BANACH SPACES

Def^n Let X be a Banach space.
A map $P: X \to X$ is a proj^n if $P^2 = P$ & $\|P\| < \infty$.

Lemma 1 Let P be a proj^n on a Banach space X.
Set $M = \text{ran}\, P$ & $N = \ker P$. Then:

(i) $M = \{x : x = Px\}$

(ii) $X = \text{span}(M, N)$

(iii) $M \cap N = \{0\}$

(iv) M & N are closed.

Proof: (iv) $M = \ker P$ is closed since P is cont
$N = \ker(I - P)$ — " — $I - P$ — " —

Lemma 2 Let X be a Banach space, and let M, N be closed linear subspaces s.t. $X = M \oplus N$.

Then $\exists$ a proj^n P s.t. $M = \operatorname{ran} P$, $N = \ker P$.

~~Sketch~~ Proof: We only need to prove that P is continuous.

Step 1 Let Y denote ~~the~~ the space X equipped w/ the norm $|||x||| = \|y\| + \|z\|$.

It is simple to prove that $|||\cdot|||$ is a norm.

That Y is complete follows from the closedness of M & N.

Step 2 Note that $P \in \mathcal{B}(Y, X)$ since $\|Px\| = \|y\| \leq \|y\| + \|z\| = |||x|||$

Step 3 ~~Prove that X &~~

That $P \in \mathcal{B}(X)$ follows if we can prove that X & Y are homeomorphic.

Consider the embedding map $J : Y \to X$.

J is obviously bijective.

J is continuous since $\|x\| = \|y + z\| \leq \|y\| + \|z\| = |||x|||$.

That J^{-1} is cont now follows from the open mapping thm.

PROJECTIONS ON HILBERT SPACES.

Def^n ~~is the~~ same as Banach space.

~~Lemma~~

Lemma 1 is identical to Banach space case.

Lemma 2 Let H be a H.S. and let M be a closed linear subspace.

$\exists$ a proj^n P s.t. $\operatorname{ran} P = M$, & $H = M \oplus \ker P$.

Proof We proved previously that $H = M \oplus M^\perp$.

~~Now apply the Banach space lemma 2~~

Given $x \in H$, we have $\underset{\in M}{y} + \underset{\in M^\perp}{z} = x$. Set $Px = y$

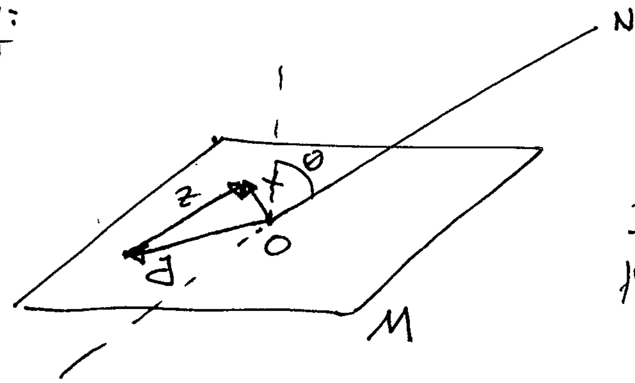
$\|Px\| = \|y\| = \sqrt{\|x\|^2 - \|z\|^2} \leq \|x\|$ so P is cont.

Def^n Let P be a proj^n on a H.S. H.
If $\operatorname{ran} P \perp \ker P$, we say that P is an orthogonal proj^n.

Lemma Let H be a H.S. and let P be a proj^n on H.
TFAE: (a) P is orthogonal
(b) $\langle Px, y\rangle = \langle x, Py\rangle \quad \forall x, y \in X$
(c) ~~$\|P\| = 1$ (if P is non-zero)~~
$\|P\| = 1$ or 0

Proof Homework

Geometry:



$Px = y$.
If N is not perpendicular to M, then $\underbrace{\|y\| \geq \|x\|}_{\exists x \text{ s.t.}}$ and so $\|P\| > 1$.
$\|P\| = \frac{1}{\cos\theta}$

In a Banach space, it is possible for $\|P\| = \infty$.