THE DUAL OF A HILBERT SPACE (review)

Let H be a H.S.

Recall that for any $y \in H$, there is an associated element $\varphi_y \in H^*$.

To be precise, set $\varphi_y(x) = (y, x)$. ← Obviously linear!

<u>Claim</u>: $\|\varphi_y\|_{H^*} = \|y\|_H$

<u>Proof</u>:

$$\left.\begin{array}{ll} |\varphi_y(x)| = |(y,x)| \leq \|y\|\,\|x\| & \Rightarrow \|\varphi_y\| \leq \|y\| \\ |\varphi_y(y)| = |(y,y)| = \|y\|^2 & \Rightarrow \|\varphi_y\| \geq \|y\| \end{array}\right\} \Rightarrow \|\varphi_y\| = \|y\|$$

In consequence, the map $H \to H^* : y \mapsto \varphi_y$ is an <u>isometry</u>.

It is trivial to verify that it is also linear.

Somewhat more ~~depp~~ deep is the fact that this map is also <u>onto</u>, <u>every</u> element of H^* takes the form of a φ_y for some y! The "Riesz Representation Theorem":

<u>Thm</u> Let H be a H.S. and let $\varphi \in H^*$.

There is a unique $y \in H$ s.t.

$$\varphi(x) = (y, x) \quad \forall x \in H.$$

ADJOINT OPERATORS ON A HILBERT SPACE

Review of "operators" (i.e. matrices) on $\mathbb{C}^n$:
Let A be an $n\times n$ matrix of complex numbers.
For any $u, v \in \mathbb{C}^n$, we have

$$(Au, v) = \sum_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{n} \overline{A_{ij} u_j} \right)}_{= \overline{[Au]_i}} v_i = \sum_{j=1}^{n} \bar{u}_j \sum_{i=1}^{n} \bar{A}_{ij} v_i = (u, \bar{A}^t v)$$

where $\bar{A}^t$ is the transpose of the complex conjugate of A.
We set $A^* = \bar{A}^t$ and call this the ADJOINT of A.

Then $\quad (Au, v) = (u, A^* v) \qquad \forall u, v \in \mathbb{C}^n$

Now let us return to the Hilbert Space case.
First we need to ask the question: Given an operator $A \in \mathcal{B}(H)$, does there necessarily exist an operator $B \in \mathcal{B}(H)$ such that

$$(Au, v) = (u, Bv) \qquad \forall u, v \in H \, ?$$

If so, then the ADJOINT of A would always exist and would be the operator B.
The following lemma answers the question affirmatively:

Lemma Let H be a H.S. and let $A \in \mathcal{B}(H)$.

Then there is a unique operator $B \in \mathcal{B}(H)$ s.t.

$$(Au, v) = (u, Bv) \quad \forall u, v \in H.$$

Moreover, $\|B\| = \|A\|$.

Proof First we define the function $B(v)$:

Fix a $v \in H$. Consider the map $\varphi(u) = \overline{(Au, v)}$.

It is easily shown that $\varphi \in H^*$ (with $\|\varphi\| \leq \|A\| \|v\|$).

By the Riesz' thm, $\exists!\, w \in H$ s.t. $\varphi(u) = (w, u)$ $\forall u$.

$\Leftrightarrow$ $\overline{(Au, v)} = (w, u)$ $\forall u$ $\Leftrightarrow$ $\cancel{(v, Au)}$ $(Au, v) = (u, w)$ $\forall u$.

We now define the map $B(v) = w$.

Is the map linear?

Pick $v_1, v_2 \in H$. What is $B(v_1 + v_2)$?

$$(Au, v_1 + v_2) = (Au, v_1) + (Au, v_2) = (u, Bv_1) + (u, Bv_2) = (u, \overbrace{Bv_1 + Bv_2}^{= B(v_1 + v_2)!})$$

We see that $B(v_1 + v_2) = Bv_1 + Bv_2$.

That $B(\lambda v) = \lambda Bv$ is shown analogously.

It only remains to prove that B is continuous (= bdd).

$$\|Bv\| = \|w\| = \|\varphi\| = \sup_{\|u\|=1} |\overline{(Au, v)}| \leq \sup_{\|u\|=1} \|A\| \|u\| \|v\| = \|A\| \|v\|$$

so $\|B\| \leq \|A\|$. To prove equality, pick $(u_n)_{n=1}^{\infty}$ such that $\|u_n\| = 1$ and $\|Au_n\| \to \|A\|$. ~~Set~~

Set $v_n = \dfrac{Au_n}{\|Au_n\|}$. Then

$$\|B\| \geq \|Bv_n\| \geq (u_n, Bv_n) = (Au_n, v_n) = \left(Au_n, \frac{Au_n}{\|Au_n\|}\right) = \|Au_n\| \to \|A\|.$$

Def^n Given $A \in \mathcal{B}(H)$ for a H.S. H,
let A^* be the operator $B \in \mathcal{B}(H)$ s.t.

$$(Au, v) = (u, A^* v) \quad \forall u, v \in H.$$

This operator is the ADJOINT of A.

Note that $A^{**} = A$. (This will not necessarily be true for unbounded operators!)

Example Consider $H = \ell^2$ with R the right-shift operator.

$$Rx = R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

For any $x, y \in H$, we now get

$$(Rx, y) = \sum_{n=2}^{\infty} \overline{x_{n-1}} y_n \underset{m=n-1}{=} \sum_{m=1}^{\infty} \overline{x_m} y_{m+1} = (x, Ly)$$

if we define L via

$$Lx = L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$ ← "left-shift operator"

so $L = R^*$ (and $L^* = R^{**} = R$)

Example Let H be a H.S., and let $(\varphi_j)_{j=1}^n$ & $(\psi_j)_{j=1}^n$ be sequences in H. Set

$$Au = \sum_{j=1}^{n} (\varphi_j, u)\psi_j$$

$$(Au, v) = \sum_{j=1}^{n} ((\varphi_j, u)\psi_j, v) = \sum_{j=1}^{n} (u, \varphi_j)(\psi_j, v) = \left(u, \sum_{j=1}^{n} \varphi_j (\psi_j, v)\right)$$

so $A^* v = \sum_{j=1}^{n} (\psi_j, v)\varphi_j$

The roles of φ_j & ψ_j have switched!

Review of linear algebra in $\mathbb{C}^n$:

Let A be an $n \times n$ matrix. Then

$$
\begin{aligned}
x \in (\operatorname{ran} A^*)^\perp &\Leftrightarrow (x, A^* y) = 0 \quad \forall y \\
&\Leftrightarrow (Ax, y) = 0 \quad \forall y \\
&\Leftrightarrow Ax = 0 \\
&\Leftrightarrow x \in \ker(A).
\end{aligned}
$$

So $\quad \ker A = (\operatorname{ran} A^*)^\perp$

take "perp" of both sides: $\quad (\ker A)^\perp = (\operatorname{ran} A^*)^{\perp\perp} = \operatorname{ran} A^*$

Set $B = A^* \Rightarrow \quad \operatorname{ran} B = (\ker A^*)^\perp$

What is useful here is that we get a classification of the range of A:

The eq^n $Ax = y$ is solvable $\Leftrightarrow$ $y \in \operatorname{ran} A \Leftrightarrow y \in (\ker A^*)^\perp$

So we only need to determine $\ker A^*$ to know when the eq^n $Ax = y$ is solvable.

This is particularly useful in infinite dimensional spaces since it often happens that $\ker A^*$ is finite dimensional, but $\operatorname{ran} A$ is infinite dimensional.

There is a complication: In an infinite dimensional space, the argument only works when $\operatorname{ran} A$ is closed.

(N.B. In $\mathbb{C}^n$, every linear space is closed!)

Lemma Let H be a H.S. and let $A \in \mathcal{B}(H)$. Then:

(a) $\ker A = (\operatorname{ran} A^*)^\perp$

(b) $\overline{\operatorname{ran} A} = (\ker A^*)^\perp$ ← Observe the closure!

(c) $X = \overline{\operatorname{ran} A} \oplus \ker A^*$

Proof (a) The calculation for $\mathbb{C}^n$ carries right over.

(b) Set $A = B^*$ in (a). Then

$$\ker B^* = (\operatorname{ran} B)^\perp \Rightarrow (\ker B^*)^\perp = (\operatorname{ran} B)^{\perp\perp} = \overline{\operatorname{ran} B}$$

(c) Just a reformulation of (b).

Corollary Let H be a H.S., and let $A \in \mathcal{B}(H)$.

(a) If $\operatorname{ran}(A)$ is closed, and $y \perp \ker(A^*)$, then the eq^n $Ax = y$ has a sol^n.

(b) If $\ker(A^*) = \{0\}$ and $\|Ax\| \geq c\|x\|$ for some $c > 0$, then $Ax = y$ has a unique sol^n for every y.

Proof (a) Obvious.

(b) Closed range theorem.

CAVEAT 1 If H is finite dimensional, and $A \in \mathcal{B}(H)$,
then $\ker A = \{0\} \Leftrightarrow \operatorname{ran} A = H$.
Not so in an infinite dimensional space.
As an example, consider $H = L^2(\mathbb{T})$,
and $\cancel{Au = x}$ $[Au](x) = x\,u(x)$.
For this operator, $\ker(A) = \{0\}$
(since if $x u(x) = 0$, then $u = 0$ a.e.).
Moreover, $A = A^*$, so $\ker A^* = \{0\}$.
The theorem says $\overline{\operatorname{ran} A} = (\ker A^*)^\perp = \{0\}^\perp = H$,
so the range is dense.
But $\operatorname{ran} A \neq H$ (for instance, $1 \in H$, but $1 \notin \operatorname{ran} A$).
(To explicitly prove that the range is dense, do this: Given $u \in H$,
set $u_n = \begin{cases} u(x) & \text{if } |x| \geq 1/n \\ 0 & \text{if } |x| < 1/n \end{cases}$ then $u_n \in \operatorname{ran} A$ and $\|u - u_n\|_{L^2} \to 0$ as $n \to \infty$)

CAVEAT 2 In finite dimensional space, $\ker A = \{0\} \Leftrightarrow \ker A^* = \{0\}$.
Not so in infinite dimensional space.
Consider for instance $H = \ell^2$ and:

$$L x = (x_2, x_3, x_4, \ldots)$$

$$\ker(L) = \operatorname{span} e_1 = \{x \in \ell^2 : x = (x_1, 0, 0, 0, \ldots)\} \neq \{0\}.$$

$L^* = R$ where

$$R x = (0, x_1, x_2, x_3, \ldots)$$

$$\ker L^* = \ker R = \{0\}$$

More review of finite-dimensional spaces.

Let H be a finite dimensional H.S. and let $A \in \mathcal{B}(H)$.

Then either the statements in the left column are all true, or the ~~column~~ statements in the right column are all true. This is known as the "Fredholm alternative".

CASE 1	CASE 2
* $\ker A = \{0\}$	* $\ker A$ is non-trivial
* $\ker A^* = \{0\}$	* $\ker A^*$ is non-trivial
* $Ax=y$ has a sol^n for every y.	* $Ax=y$ has a sol^n only if $y \in (\ker A^*)^\perp$
~~* No sol^n to $Ax=y$ is unique.~~	* No sol^n to $Ax=y$ is unique
* $Ax=y$ has no more than one sol^n	

Such a dichotomy does not hold for all operators on a infinite dimensional H.S., as we have seen.

However, there are classes of operators that do admit such a binary classification. Consider for instance the class of operators of the form $A = I + K$, where K is a compact operator. Any such operator satisfies:

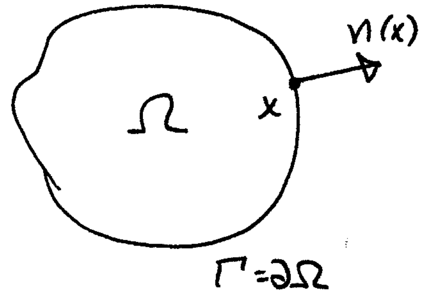
(1) $\mathrm{ran}(A)$ is closed

(2) $\dim(\mathrm{ran}\,A)$ and $\dim(\mathrm{ran}\,A^*)$ are equal and finite.

Therefore, ~~A~~ this class satisfies the Fredholm alternative.

<u>Example</u> Consider the PDE

$$(\text{PDE}) \quad \begin{cases} -\Delta u = 0 & \text{on } \Omega \\ \dfrac{\partial u}{\partial n} = f & \text{on } \Gamma = \partial\Omega \end{cases}$$



We write the sol^n in the form

$$u(x) = \int_\Gamma G(x,y)\sigma(y)\,ds(y) \quad \text{where } G(x,y) = \log|x-y|$$

Then for $x \in \Gamma$ we have (check for factor 2π!)

$$\frac{\partial u}{\partial n}(x) = n(x)\cdot \lim_{x' \to x} \nabla u(x') = \lim_{x'\to x} n(x)\cdot \int_\Gamma \frac{x'-y}{|x'-y|^2}\sigma(y)\,ds(y) = \frac{1}{2}\sigma(x) + \int_\Gamma \frac{n(x)\cdot(x-y)}{|x-y|^2}\sigma(y)\,ds(y)$$

So (PDE) is satisfied iff

$$(\text{BIE}) \quad \left(\tfrac{1}{2}I + D\right)\sigma = f$$

where

$$[D\sigma](x) = \int_\Gamma \frac{n(x)\cdot(x-y)}{|x-y|^2}\sigma(y)\,ds(y). \quad \leftarrow \text{compact op. on } L^2(\Gamma)$$

We next determine D^* (in $L^2(\Gamma)$):

$$(D\sigma,\eta)_{L^2(\Gamma)} = \int_\Gamma \int_\Gamma \frac{n(x)\cdot(x-y)}{|x-y|^2}\sigma(y)\,ds(y)\,\eta(x)\,ds(x) =$$

$$= \int_\Gamma \sigma(y) \underbrace{\int \frac{n(x)\cdot(x-y)}{|x-y|^2}\eta(x)\,ds(x)}_{=[D^*\eta](y)}\,ds(y) = (\sigma, D^*\eta)_{L^2(\Gamma)}$$

so $\displaystyle [D^*\sigma](x) = \int_\Gamma \frac{n(y)\cdot(x-y)}{|x-y|^2}\sigma(y)\,ds(y)$

We now find that (BIE) has a sol^n iff $f \in \ker\left(\tfrac{1}{2}I + D^*\right)^\perp = \operatorname{span}\{1\}^\perp$.

We recover the well-known result that (PDE) is well posed iff $\int_\Gamma f(x)\,ds(x) = (1,f) = 0$

SELF-ADJOINT OPERATORS

Def^n Let H be a H.S.

An operator $A \in \mathcal{B}(H)$ is SELF-ADJOINT if $A^* = A$.

In other words, A is S-A if $(Ax, y) = (x, Ay)$ $\forall x, y \in H$.

Given a S-A operator A, we define a "bilinear form"

$a(x,y) = (Ax, y) = (x, Ay)$. Strictly speaking

Note that $a(x,y) = \overline{a(y,x)}$ (which we call "Hermitian symmetric")

and in consequence $a(x,x) \in \mathbb{R}$.

This allows us to classify S-A operators:

(*) A is NON-NEGATIVE if $(x, Ax) \geq 0$ $\forall x$

(*) A is POSITIVE if $(x, Ax) > 0$ $\forall x$

(*) A is COERCIVE if $\exists c > 0$ s.t. $(x, Ax) \geq c\|x\|^2$ $\forall x$

The terms "non-positive" and "negative" are defined analogously.

Examples

* Any projection that is S-A is non-negative.
* The operator $[Au](x) = x\,u(x)$ on $L^2(\mathbb{I})$ is positive, but not coercive.
* If B is S-A and $\|B\| < 1$, then $A = I + B$ is coercive with $c = 1 - \|B\|$.

Now suppose that A is a bounded and coercive operator.

Set $\|x\|_A = \sqrt{(x, Ax)}$ ← Definition of a new norm.

Then
$$c\|x\| \leq \|x\|_A \leq \sqrt{\|A\|}\;\|x\|,$$
in other words, the new norm $\|\cdot\|_A$ is equivalent to the standard norm.

The following is a "baby version" of the famous "Lax-Milgram" theorem:

<u>Thm</u> Let H be a H.S. and let $A \in \mathcal{B}(H)$ be S-A, and coercive.
Then for any $y \in H$, the eq^n
$$Ax = y$$
has a unique sol^n.

<u>Proof</u> Set $\varphi(z) = (y, z)$.
Let H_A denote the H.S. with inner product
$$(x, y)_A = (Ax, y)$$
Then $\|x\|_A^2 = (Ax, x) \geq c\|x\|^2 \Rightarrow \|x\| \leq \frac{1}{\sqrt{c}}\|x\|_A$.
Now observe that
$$|\varphi(z)| = |(y, z)| \leq \|y\|\,\|z\| \leq \frac{1}{\sqrt{c}}\|y\|\,\|z\|_A.$$
Consequently, $\varphi \in H_A^*$.
By the Riesz repr^n thm, $\exists!\, x \in H = H_A$ s.t.
$$\varphi(z) = (x, z)_A \quad \forall z.$$
But then
$$\underbrace{(y, z)}_{=\varphi(z)} = \underbrace{(Ax, z)}_{=(x, z)_A} \quad \forall z$$
which is to say that $y = Ax$.

Let H be a H.S., and set $B_1 = \{x \in H : \|x\| \le 1\}$ ← unit ball.

For any $A \in \mathbb{B}(H)$, we have

$$\|A\| = \sup_{x \in B_1} \|Ax\| = \sup_{x \in B_1} \sup_{y \in B_1} (Ax, y) = \sup_{x,y \in B_1} (Ax, y). \qquad (1)$$

For a S-A operator, we ~~can~~ have the much better equality:

<u>Lemma</u> Let H be a H.S., let $A \in \mathbb{B}(H)$ be S-A. Then $\|A\| = \sup_{x \in B_1} |(Ax, x)|$

<u>Proof</u> Set $\varphi(x) = (Ax, x)$, and $\alpha = \sup_{x \in B_1} |\varphi(x)|$.

We have $|\varphi(x)| = |(Ax, x)| \le \|Ax\| \|x\| \le \|A\| \|x\|^2$, so obviously $\alpha \le \|A\|$.

To prove $\alpha \ge \|A\|$, we first note that for any $x, y \in H$ and $\beta \in \mathbb{C}$,

$$\varphi(x+\beta y) - \varphi(x - \beta y) = (Ax + \beta Ay, x + \beta y) - (Ax - \beta Ay, x - \beta y) =$$

$$= \cdots = 2(Ax, \beta y) + 2(\beta Ay, x) \stackrel{*}{=} \text{ use that } A \text{ is S-A!}$$

$$= 2(Ax, \beta y) + 2(\beta y, Ax) \qquad (2)$$

Now pick β so that $|\beta| = 1$ and $(Ax, \beta y) \in \mathbb{R}$,

then $(\beta y, Ax) = \overline{(Ax, \beta y)} = (Ax, \beta y)$ so from (2) we get

$$\varphi(x + \beta y) - \varphi(x - \beta y) = 4(Ax, \beta y).$$

Since $|\beta| = 1$, we then find

$$|(Ax, y)| = |(Ax, \beta y)| = \left|\tfrac{1}{4}\left(\varphi(x+\beta y) - \varphi(x - \beta y)\right)\right| \le \tfrac{1}{4}\left(|\varphi(x+\beta y)| + |\varphi(x - \beta y)|\right)$$

def^n of α → $$\le \tfrac{1}{4}\left(\alpha \|x + \beta y\|^2 + \alpha \|x - \beta y\|^2\right) = \cdots = \tfrac{1}{2}\left(\alpha \|x\|^2 + \alpha \|y\|^2 \underbrace{|\beta|^2}_{=1}\right) \qquad (3)$$

Now use the general identity (1):

$$\|A\| = \sup_{x,y \in B_1} |(Ax, y)| \underset{\text{Use (3)}}{\le} \sup_{x,y \in B_1} \frac{\alpha}{2}\left(\|x\|^2 + \|y\|^2\right) = \alpha$$

Done!

Def^n Let B be an operator on a H.S.
We say that B is SKEW-ADJOINT if $B^* = -B$

Lemma B is skew-adjoint $\Leftrightarrow$ $B = iA$ for some S-A operator A.

Example $H = \mathbb{C}^n$. ~~Let U be a unitary $n\times n$ matrix~~ ~~in other words,~~

Set $D = \begin{bmatrix} \lambda_1 & 0 & 0 & - \\ 0 & \lambda_2 & 0 & - \\ 0 & 0 & \lambda_3 & - \\ | & | & | & \end{bmatrix} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are some complex numbers.

Then $D^* = \bar{D}^t = \bar{D} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n)$

We find that if all λ_j's are real (i.e. $\text{Im}(\lambda_j) = 0$), then $D^* = D$ an so D is S-A.

Conversely, if all λ_j's are purely imaginary, then $D^* = \bar{D} = -D$ so D is skew-adjoint.

Let X be an arbitrary $n\times n$ matrix and set

$A = X D X^*$ & D is defined as above

Then $A^* = X^{**} D^* X^* = X \bar{D} X^*$.

So again: A is self-adjoint $\Leftrightarrow$ $\text{Im}(\lambda_j) = 0 \ \forall j$

A is skew-adjoint $\Leftrightarrow$ $\text{Re}(\lambda_j) = 0 \ \forall j$