SELF-ADJOINT OPERATORS

Thm Let H be a H.S., and let $A \in \mathcal{B}(H)$ be S-A. Then:

(a) $\sigma_p(A) \subseteq \mathbb{R}$

(b) If $\lambda, \mu \in \sigma_p(A)$ and $\lambda \neq \mu$, then $\ker(A-\lambda I) \perp \ker(A-\mu I)$

Proof (a) Suppose $Ax = \lambda x$ and $x \neq 0$. Then

$$\lambda \|x\|^2 = \lambda (x,x) = (x, \lambda x) = (x, Ax) = (Ax, x) = (\lambda x, x) = \bar{\lambda} \|x\|^2$$

(b) Suppose $Ax = \lambda x$, $Ay = \mu y$, and $\lambda \neq \mu$. Then

$$\lambda(x,y) \underset{\lambda \in \mathbb{R}}{=} (\lambda x, y) = (Ax, y) = (x, Ay) = (x, \mu y) = \mu(x,y)$$

so $\underbrace{(\lambda - \mu)}_{\neq 0}(x,y) = 0$

Thm Let H be a H.S, and let $A \in \mathcal{B}(H)$ be S-A. Then

(a) $\sigma(A) \subseteq \mathbb{R}$

(b) $\sigma_r(A) = \emptyset$

Lemma Let H be a H.S., and let $A \in \mathcal{B}(H)$.
Then if $\lambda \in \sigma_r(A)$, then $\bar{\lambda} \in \sigma_p(A^*)$.

Proof of lemma Suppose $\lambda \in \sigma_r(A)$.

Then $\overline{\operatorname{ran}(A-\lambda I)} \neq H$ so $\exists\ x \in \operatorname{ran}(A-\lambda I)^\perp$ s.t. $x \neq 0$.

Now $\operatorname{ran}(A-\lambda I)^\perp = \ker(A^* - \bar{\lambda} I)$ so $A^* x = \bar{\lambda} x$.

Proof of thm

(a) Suppose $\lambda = a + ib$ with $b \neq 0$. Then

$$\|(A-\lambda I)x\|^2 = \|(A-aI)x - ibx\|^2 =$$

$$= \underbrace{\|(A-aI)x\|^2}_{\geq 0} - 2\,\mathrm{Re}\Big[\underbrace{((A-aI)x, ibx)}_{= i\underbrace{((A-aI)x,\, bx)}_{\in \mathbb{R}}}\Big] + \underbrace{\|ibx\|^2}_{= b^2\|x\|^2} \geq b^2\|x\|^2.$$

Since $A-\lambda I$ is coercive, $\ker(A-\lambda I) = \{0\}$ so $\lambda \notin \sigma_p(A)$.
Moreover $A-\lambda I$ has closed range, so $\lambda \notin \sigma_c(A)$.
Finally, if λ were to be in $\sigma_r(A)$, then $\bar{\lambda} = a - ib \in \sigma_p(A^*) = \sigma_p(A)$ which is impossible since $\sigma_p(A) \subseteq \mathbb{R}$.

(b) Suppose that $\lambda \in \sigma_r(A)$. Then $\bar{\lambda} \in \sigma_p(A^*) = \sigma_p(A)$.
Therefore $\lambda = \bar{\lambda}$ and $\lambda \in \sigma_r(A) \cap \sigma_p(A) = \emptyset$.

Def^n Let H be a H.S, and let $A \in \mathcal{B}(H)$.
For $\lambda \in \sigma_p(A)$, define the <u>multiplicity</u> of λ as $\dim(\ker(A - \lambda I))$

Note In a H.S., the multiplicity may in general be infinite.
As an example, consider $A = I$ and $\lambda = 1$.
Then $\ker(A - \lambda I) = \ker(I - I) = \ker(0) = H$.

Thm Let H be a H.S., and let $A \in \mathcal{B}(H)$ be compact & S-A. Then

(a) If $\lambda \in \sigma_p(A)$ and $\lambda \neq 0$, then λ has finite multiplicity

(b) If $\sigma_p(A)$ is infinite, then 0 is an accumulation point of $\sigma_p(A)$, and there are no other accumulation points.

Note: The thm implies that the non-zero evals of A can be ordered so that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ and that $|\lambda_n| \to 0$ as $n \to \infty$.

Proof: (a) By contradiction.

Suppose $\lambda \in \sigma_p(A)$, $\lambda \neq 0$, and $\dim(\ker(A - \lambda I)) = \infty$.

Then $\exists$ an ON-seq $(e_n)_{n=1}^{\infty}$ s.t. $Ae_n = \lambda e_n$.

$e_n \rightharpoonup 0$, and A is compact so $Ae_n \to 0$.

But this is impossible since $\|Ae_n - 0\| = \|\lambda e_n - 0\| = |\lambda|$

(b) Suppose that $\sigma_p(A)$ is infinite.

Since $\sigma_p(A)$ is bdd ($|\lambda| \leq \|A\|$ when $\lambda \in \sigma(A)$) there must be at least one accumulation point λ.

We will prove that if $\lambda \neq 0$, then λ cannot be an acc. point.

Suppose $\lambda \neq 0$. Then we can pick $\lambda_n \in \sigma_p(A)$ s.t.

$\lambda_n \to \lambda$, $|\lambda_n| \geq \frac{|\lambda|}{2}$ $\forall n$, and $\lambda_n \neq \lambda_m$ when $n \neq m$.

Let e_n be s.t. $Ae_n = \lambda_n e_n$.

Set $f_n = \frac{1}{\lambda_n} e_n$. Then $\|f_n\| = \frac{1}{|\lambda_n|} \leq \frac{2}{|\lambda|}$ so we can pick a convergent subseq $f_{n_j} \rightharpoonup f$.

Since A is compact, $Af_{n_j} \to Af$.

This is impossible since $Af_{n_j} = A\frac{1}{\lambda_{n_j}} e_{n_j} = e_{n_j}$.

Alternative end: Set $\Omega = \{\frac{1}{\lambda_n} e_n\}_{n=1}^{\infty}$

Ω bdd so $A\Omega$ compact

$A\Omega = \{e_n\}_{n=1}^{\infty}$ which is impossible!

Lemma Let H be H.S., and let $A \in \mathcal{B}(H)$ be S-A and compact. Then either $\|A\|$, or $-\|A\|$, or both, belong to $\sigma_p(A)$

Proof Recall that $\|A\| = \sup_{\|u\|=1} |(Au, u)|$

Therefore, there is a seq $(u_n)_{n=1}^{\infty}$ s.t. $\|u_n\| = 1$, and $(Au_n, u_n) \to \lambda$ where $\lambda = \pm\|A\|$.

(u_n) bdd $\Rightarrow \exists$ subseq $(u_{n_j})_{j=1}^{\infty}$ s.t. $u_{n_j} \rightharpoonup u$.

Since A is compact $Au_{n_j} \to Au =: v$.

We will prove that $Av = \lambda v$:

$$\|(A - \lambda I) v\|^2 = \lim_{j\to\infty} \|(A - \lambda I) A u_{n_j}\|^2 \le \|A\|^2 \lim_{j\to\infty} \|(A - \lambda I) u_{n_j}\|^2 =$$

$$= \|A\|^2 \lim_{j\to\infty} \Big[\underbrace{\|Au_{n_j}\|^2}_{\le \lambda^2} - 2\lambda \underbrace{(Au_{n_j}, u_{n_j})}_{\to \lambda} + \lambda^2 \underbrace{\|u_{n_j}\|^2}_{=1} \Big] = 0$$

It only remains to prove that $v \ne 0$.

Suppose $v = 0$. Then $\|Au_{n_j}\| \to 0$, whence

$$\|A\| = |\lambda| = \lim_{j\to\infty} |(Au_{n_j}, u_{n_j})| \le \limsup_{j\to\infty} \|Au_{n_j}\| = 0$$

Invariant subspaces

Let H be a H.S, and suppose $H = M \oplus N$ where $M = N^\perp$. ~~The Pe~~ Let P & Q denote orthog proj^{ns} onto M & N.

Write $x \sim \begin{bmatrix} Px \\ Qx \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Now let $A \in \mathcal{B}(H)$. We have, for $y = Ax$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} PAx \\ QAx \end{bmatrix} = \begin{bmatrix} PAPx + PAQx \\ QAPx + QAQx \end{bmatrix} = \begin{bmatrix} PAP & PAQ \\ QAP & QAQ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now suppose that M & N are <u>invariant subspaces</u> of A which means that $AM \subseteq M$ (i.e. $Ax \in M$, whenever $x \in M$)
$AN \subseteq N$ (i.e. $Ax \in N$, whenever $x \in N$).

Then $PAQ = 0$ & $QAP = 0$ so

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} PAP & 0 \\ 0 & QAQ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ ← the operator is <u>block diagonal</u>.

Now, if A is S-A, and $AM \subseteq M$, then $AM^\perp \subseteq M^\perp$ automatically!

<u>Lemma</u> Suppose H is a H.S. and that $A \in \mathcal{B}(H)$ is S-A. Then if M is an invariant subspace of A, so is $M^\perp$.

<u>Proof</u> Suppose $x \in M^\perp$.
$\forall y \in M$: $(Ax, y) = (x, Ay) = 0$ since $x \in M^\perp$ and $Ay \in M$.

<u>Note</u> The lemma is not true for general ops:
$H = \mathbb{C}^2$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $M = \mathrm{span}(e_1)$ ← invariant
$M^\perp = \mathrm{span}(e_2)$ ← not invariant.

Suppose A is S-A, and $Av = \lambda v$ for some $v \neq 0$.

Then $M = \text{span}(v)$ is an invariant subspace.

$(x \in M \Rightarrow x = \alpha v \Rightarrow Ax = \alpha Av = \alpha \lambda v \in M.)$

Let P denote proj^n onto M.

Then $APx = \lambda Px$ so $AP = \lambda P$.

<u>Example</u> $H = \mathbb{C}^n$ $A \in \mathcal{B}(H)$ is S-A.

Then A has an ON-basis $(e_j)_{j=1}^n$ s.t. $Ae_j = \lambda_j e_j$.

Let P_j denote orthog proj^n onto span(e_j).

Then $AP_j x = \lambda_j P_j x$ and

$$A = A \sum_{j=1}^n P_j = \sum_{j=1}^n \lambda_j P_j = \sum_{j=1}^n \lambda_j e_j e_j^*$$

<u>Thm</u> Let H be a H.S., and let $A \in \mathcal{B}(H)$ be compact and S-A.

Then there is an ON-seq $(e_n)_{n=1}^N$ (N may be infinite) s.t.

* $Ae_n = \lambda_n e_n$
* $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$
* If $N = \infty$, then $|\lambda_n| \to 0$
* $A = \sum_{n=1}^N \lambda_n P_n$ where $P_n x = e_n(e_n, x)$ and the sum converges in norm if $N = \infty$.

Moreover, if $\ker(A) = \{0\}$, then $\{e_n\}_{n=1}^N$ is an ON-basis for H.

If $\ker(A) \neq \{0\}$, and if $\{f_m\}_{m=1}^M$ is an ON-basis for $\ker(A)$, then $(e_n)_{n=1}^N \cup (f_m)_{m=1}^M$ is an ON-basis for H.

<u>Proof</u> First we construct subspaces (M_n) and (N_n), and operators (A_n) via the following procedure:

<u>Step 1</u> Set $N_1 = H$, ~~and~~ $M_1 = \{0\}$, and $A_1 = A$.
$\exists\, \lambda_1$ and e_1 s.t. $Ae_1 = \lambda_1 e_1$, $\|e_1\| = 1$, and $|\lambda_1| = \|A_1\|$.
Set P_1 = proj^n onto span(e_1).

<u>Step 2</u> Set $M_2 = \text{span}(e_1)$ and $N_2 = M_2^{\perp}$,
and let A_2 denote the restriction of A to N_2, $A_2 = A_1 - \lambda_1 P_1$
$\exists\, \lambda_2$ and e_2 s.t. $Ae_2 = \lambda_2 e_2$, $\|e_2\| = 1$, and $|\lambda_2| = \|A_2\|$
Set P_2 = orthog proj^n onto span(e_2).

$\vdots$

<u>Step n</u> Set $M_n = \text{span}(e_1, e_2, \dots, e_{n-1})$ and $N_n = M_n^{\perp}$,
and let A_n denote the restriction of A to N_n, $A_n = A - \sum_{j=1}^{n-1} \lambda_j P_j$
$\exists\, \lambda_n$ and e_n s.t. $Ae_n = \lambda_n e_n$, $\|e_n\| = 1$, and $|\lambda_n| = \|A_n\|$
Set P_n = orthog proj^n onto span(e_n).

Note that at the n'th step, $A = \sum_{j=1}^{n} \lambda_j P_j + A_{n+1}$

Proof cont'd

The process may end in two ways:

Case 1 For some n, $A_{n+1}=0$.

In this case A has finite rank, $A=\sum_{j=1}^{n} \lambda_j P_j$

$H=\text{span}(e_1,e_2,\dots,e_n)\oplus \ker(A)$

Let $(f_m)_{m=1}^{M}$ be an ON-basis for $\ker(A)$.

Case 2 $A_n \neq 0 \quad \forall n$.

Then $\|A - \sum_{j=1}^{n} \lambda_j P_j\| = \|A_{n+1}\| = |\lambda_{n+1}| \to 0$ as $n\to\infty$.

So $A=\sum_{n=1}^{\infty} \lambda_n P_n$

However, (e_n) is not necessarily a basis.

If $\overline{\text{span}(e_n)}=H$, then it is, and we are done.

If not, then suppose $x\in \text{span}(e_n)^{\perp}$ and $x\neq 0$.

Then $x\in N_n \quad \forall n$ so

$\|Ax\|=\|A_n x\| \le \|A_n\|\,\|x\| \to 0$ as $n\to\infty$

So $x\in\ker(A)$.

Thus $H=\overline{\text{span}(e_n)}\oplus\ker(A)$.