APPM5450 — Applied Analysis: Final exam — Solutions

7:30 – 9:50, May 9, 2013. Closed books.

Problem 1: (12p) No motivation required for these problems.

(a) (3p) Let $n \in \mathbb{Z}$ and define $f_n \in \mathcal{S}^*(\mathbb{R})$ via $f_n(x) = \sin(nx)$. What is $\hat{f}_n$?

(b) (3p) State for which $p \in [1, \infty]$, if any, the unit ball in $L^p(\mathbb{R})$ is weakly compact.

(c) (3p) Set $H = L^2(\mathbb{R})$ and define $T \in \mathcal{B}(H)$ via $[Tu](x) = u(-x)$. What is $\sigma(T)$?

(d) (3p) Let H be a Hilbert space. State the definition of a *unitary* operator on H.

Solution

(a) Observe that
- $\sin(nx) = (1/2i)(e^{inx} - e^{-inx})$,
- $\mathcal{F}\delta = \beta$ (where $\beta = 1/\sqrt{2\pi}$),
- $[\mathcal{F}(e^{inx}g)](t) = \hat{g}(t-n)$.

Combining, we find

$$\hat{f}_n(t) = (1/2i)\big(\beta\,\delta(t-n) - \beta\,\delta(t+n)\big) = \frac{i}{2\sqrt{2\pi}}\,\delta(t+n) - \frac{i}{2\sqrt{2\pi}}\,\delta(t-n).$$

(b) This is Banach-Alaoglu, which applies in reflexive spaces. Consequently, the unit ball is weakly compact when $p \in (1, \infty)$.

(c) Observe that T is both unitary and self-adjoint. This means that the spectrum is contained in the intersection of the real line and the unit circle, which is to say $\sigma(T) \subseteq \{-1, 1\}$. It is then easily verified that any even function is a eigenvector associated with $\lambda = 1$ and any odd function is an eigenvector associated with $\lambda = -1$. So $\sigma(T) = \sigma_{\rm p}(T) = \{-1, 1\}$.

(d) A unitary operator is a <u>bijective operator</u> that <u>preserves the inner product</u>.

Problem 2: (13p) Let H be a Hilbert space, and let A denote a bounded linear operator on H.

(a) (3p) State the definition of the *resolvent set* $\rho(A)$ of A.

(b) (10p) Prove that the resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$.

Solution

(a) $\rho(A)$ is the set of complex numbers λ such that $A - \lambda\,I$ is one-to-one and onto.

(b) Fix $\lambda \in \rho(A)$. Then $A - \lambda I$ is continuously invertible by the open mapping theorem. Set $\varepsilon = 1/||(A - \lambda I)^{-1}||$ and observe that $\varepsilon > 0$. For any $\mu \in B_\varepsilon(\lambda)$, we find

$$A - \mu\,I = A - \lambda I - (\mu - \lambda)\,I = (A - \lambda I)\,\big[I - (\mu - \lambda)\,(A - \lambda I)^{-1}\big]. \tag{1}$$

Now observe that

$$||(\mu - \lambda)\,(A - \lambda I)^{-1}|| \le |\mu - \lambda|\,||(A - \lambda\,I)^{-1}|| < \varepsilon\,||(A - \lambda I)^{-1}|| = 1.$$

Consequently, the Neumann series argument shows that the expression in brackets in (1) is invertible.

Problem 3: (16p) Define for $\alpha, \beta \in (0, \infty)$ and for $n = 1, 2, 3, \dots$ functionals A_n, $B_n \in \mathcal{S}^*(\mathbb{R})$ via

$$A_n(\varphi) = \sum_{j=1}^{n} \alpha^j \, \varphi(j), \qquad \text{and} \qquad B_n(\varphi) = \sum_{j=1}^{n} j^\beta \, \varphi(j).$$

(a) (8p) For which $\alpha \in (0, \infty)$ does the sequence $(A_n)_{n=1}^\infty$ converge in $\mathcal{S}^*(\mathbb{R})$?

(b) (8p) For which $\beta \in (0, \infty)$ does the sequence $(B_n)_{n=1}^\infty$ converge in $\mathcal{S}^*(\mathbb{R})$?

Solution

Answer: For $\alpha \in (0, 1]$ and for any $\beta \in (0, \infty)$.

To prove that, e.g., (B_n) converges, we need to show that for every $\varphi \in \mathcal{S}$, the sequence $\big(B_n(\varphi)\big)_{n=1}^\infty$ converges to some number $B(\varphi)$, where $B \in \mathcal{S}^*$.

To prove that (A_n) converges, we will show that there exists a $\varphi \in \mathcal{S}$, such that the sequence $\big(B_n(\varphi)\big)_{n=1}^\infty$ diverges..

- **Case 1:** $\beta \in (0, \infty)$

  Pick k such that $k > \beta + 1$. Then

  $$|B_n(\varphi)| \le \sum_{j=1}^{\infty} j^\beta |\varphi(j)| \le \sum_{j=1}^{\infty} j^\beta \frac{||\varphi||_{0,k}}{(1 + j^2)^{k/2}} \sim ||\varphi||_{0,k} \sum_{j=1}^{\infty} j^{\beta - k} < \infty.$$

- **Case 2:** $\alpha \in (0, 1]$

  The proof is entirely analogous to Case 1 since the "weights" are bounded:

  $$|A_n(\varphi)| \le \sum_{j=1}^{\infty} |\varphi(j)| \le \sum_{j=1}^{\infty} \frac{||\varphi||_{0,2}}{1 + j^2} \le C \, ||\varphi||_{0,2}.$$

- **Case 3:** $\alpha \in (1, \infty)$

  Note that the weights grow exponentially in this case, which means that we cannot dominate the sum using a polynomial decay factor. We instead seek a Schwartz function φ such that $\alpha^j \varphi(j) \to \infty$. To this end, pick $\gamma \in (1, \alpha)$, and set

  $$\varphi(x) = \gamma^{-x^2/\sqrt{1+x^2}}.$$

  Then $\varphi \in \mathcal{S}(\mathbb{R})$, but

  $$A_n(\varphi) = \sum_{j=1}^{n} \alpha^j \, \varphi(j) \sim \sum_{j=1}^{n} \alpha^j \gamma^{-j} = \sum_{j=1}^{n} (\alpha/\gamma)^j \to \infty.$$

Problem 4: (23p) Let $\mathbb{T}$ denote the unit circle as usual, and define a function $f \in L^2(\mathbb{T})$ via $f(x) = x$, where $\mathbb{T}$ is parameterized using $x \in [-\pi, \pi)$.

(a) (5p) What are the Fourier coefficients of f?

(b) (5p) For which $s \in [0, \infty)$ is it the case that $f \in H^s(\mathbb{T})$?

(c) (5p) Use your result in (a) to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

(d) (5p) Let g denote the real-valued function obtained via periodic continuation of f to a 2π periodic function on $\mathbb{R}$. Prove that $g \in \mathcal{S}^*(\mathbb{R})$.

(e) (3p) What is the Fourier transform of the function $g \in \mathcal{S}^*(\mathbb{R})$ defined in (d)? No motivation required for this part. (Hint: Problem 1(a) may be useful.)

Solution

(a) Set $\beta = 1/\sqrt{2\pi}$. Then $\alpha_n = \beta \int_{-\pi}^{\pi} e^{-inx}\, x\, dx = \beta i \int_{-\pi}^{\pi} \sin(nx)\, x\, dx = \cdots = \frac{2\beta i \pi (-1)^n}{n}$.

(b) We find $||f||_{H^s}^2 = \sum (1 + |n|^2)^s |\alpha_n|^2 = \sum (1 + |n|^2)^s \frac{4\beta^2\pi^2}{n^2} \sim \sum n^{2s}\, n^{-2}$.

The sum is finite iff $2s - 2 < -1$, which is to say $s < 1/2$.

(c) Parseval's theorem states that $||f||_{L^2}^2 = \sum |\alpha_n|^2$. Now

$$\sum_{n=-\infty}^{\infty} |\alpha_n|^2 = 2\sum_{n=1}^{\infty} \frac{4\beta^2\pi^2}{n^2} = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2},$$

$$||f||_{L^2}^2 = \int_{-\pi}^{\pi} x^2\, dx = 2\int_0^{\pi} x^2\, dx = (2/3)\pi^3.$$

(d) For a given $\varphi \in \mathcal{S}$, we can bound T_f as follows:

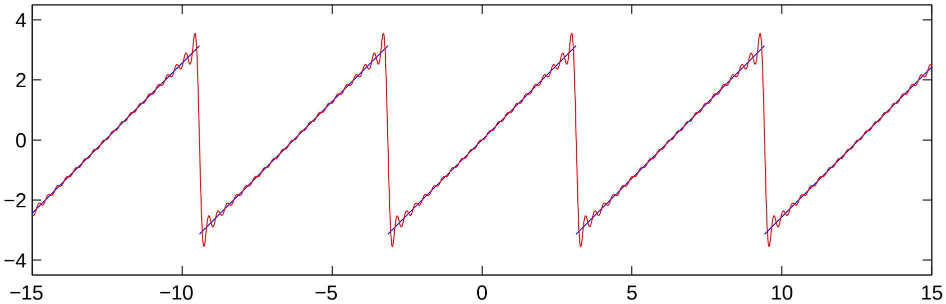
$$|T_f(\varphi)| = \left| \int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx \right| \leq \int_{-\infty}^{\infty} |f(x)| \frac{||\varphi||_{0,2}}{1 + x^2}\, dx \leq \int_{-\infty}^{\infty} \pi \frac{||\varphi||_{0,2}}{1 + x^2}\, dx = \pi^2 ||\varphi||_{0,2}.$$

(e) We have $f(x) = \sum_{n=-infty}^{\infty} \alpha_n \beta\, e^{inx}$. Since $[\mathcal{F}e^{inx}](t) = \beta\, \delta(t - n)$, we get

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} \alpha_n \beta^2 \delta(t - n) = \sum_{n=-\infty}^{\infty} \frac{2\beta i \pi (-1)^n}{n} \beta^2 \delta(t - n) = \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{n\sqrt{2\pi}} \delta(t - n).$$

We treated the sum in a cavalier manner, but we only needed the answer!

Note: The Fourier sum simplifies as $f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$. The first 20 terms look like:



Problem 5: (18p) Set $I = (0, 1)$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of Lebesgue integrable real valued functions on the interval $I = (0,\, 1)$ such that for every $x \in I$,

$$\lim_{n\to\infty} f_n(x) = x.$$

Consider for $n = 1,\, 2,\, 3,\, \ldots$ the three sequences

$$a_n = \int_0^1 f_n(x)\, dx$$

$$b_n = \int_0^1 \frac{f_n(x)}{1 + (f_n(x))^2}\, dx$$

$$c_n = \int_0^1 \Big|\sum_{j=1}^{n} f_j(x)\Big|\, dx.$$

Which of the sequences must necessarily converge as $n \to \infty$? Is it for any of the convergent sequences possible to say what the limit is? Motivate your answers.

Solution

The sequence a_n: This may or may not converge.
If say $f_n(x) = x$ for all x, then $a_n \to 1/2$.
If on the other hand $f_n = n^2\, \chi_{(0,\,1/n)} + x\, \chi_{(1/n,\,1)}$, then $a_n \to \infty$.

The sequence b_n: The absolute value of the integrand is bounded by $g(x) = 1$. Since $\int_0^1 g\, dx = 1$ is finite, Lebesgue dominated convergence applies and we find that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \int_0^1 \frac{f_n(x)}{1 + (f_n(x))^2}\, dx = \int_0^1 \lim_{n\to\infty} \frac{f_n(x)}{1 + (f_n(x))^2}\, dx = \int_0^1 \frac{x}{1 + x^2}\, dx$$

$$= \left[\frac{1}{2}\log(1 + x^2)\right]_0^1 = \frac{1}{2}\big(\log(2) - \log(1)\big) = \log(2)/2.$$

The sequence c_n: Since the integrand is non-negative, Fatou's lemma applies:

$$\liminf_{n\to\infty} c_n = \liminf_{n\to\infty} \int_0^1 \Big|\sum_{j=1}^{n} f_j(x)\Big|\, dx \geq \int_0^1 \liminf_{n\to\infty} \Big|\sum_{j=1}^{n} f_j(x)\Big|\, dx.$$

For any x, we have $\sum_{j=1}^{n} f_j(x) \to \infty$, so $\liminf_{n\to\infty} \Big|\sum_{j=1}^{n} f_j(x)\Big| = \infty$, and consequently $c_n \to \infty$.

Problem 6: (18p) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^2(\mathbb{R})$ that converges pointwise to a function f. In other words,

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for all } x \in \mathbb{R}.$$

Suppose further that all f_n satisfy

$$|f_n(x)| \leq 2|f(x)|, \qquad \text{for all } x \in \mathbb{R}.$$

For each of the three sets of conditions on f given below, specify for which $r \in [1, \infty)$ it is necessarily the case that

$$\lim_{n\to\infty} ||f - f_n||_{L^r(\mathbb{R})} = 0.$$

(a) (6p) $f \in L^2(\mathbb{R})$, and for $|x| \geq 2$, it is the case that $f(x) = 0$.

(b) (6p) $f \in L^2(\mathbb{R})$ and $|f(x)| \leq 2$ for all $x \in \mathbb{R}$.

(c) (6p) $f \in L^2(\mathbb{R})$ and $f \in L^3(\mathbb{R})$.

Solution

Answers: (a) $r \in [1,\, 2]$. (b) $r \in [2,\, \infty)$. (c) $r \in [2, 3]$.

We need to prove the claim when it is true, and provide counter-examples when it is not. The basic question we need to resolve is when

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - f_n(x)|^r \, dx = 0. \tag{2}$$

The integrand in (2) converges to zero pointwise, and we want to bring the LDCT to bear. To this end, we construct a dominator h via

$$|f(x) - f_n(x)|^r \leq \big(|f(x)| + |f_n(x)|\big)^r \leq \big(|f(x)| + 2\,|f(x)|\big)^r = 3^r\,|f(x)|^r =: h(x).$$

We will analyze each of the three assumptions to see when $\int h < \infty$.

(a) Suppose $r \in [1, 2]$. Then $h(x) = 3^r\,|f(x)|^r \leq 3^r \max(1, |f(x)|^2)$. Since $f \in L^2$, and since in this case, h has compact support, we find $\int h < \infty$.

Suppose $r > 2$. When $|f(x)| > 1$, we have $|f(x)|^r > |f(x)|^2$, so h does not necessarily have finite integral and the LDCT does not apply. We look for a counter-example. Pick a real number α such that $-\frac{1}{2} < \alpha < -\frac{1}{r}$, and set $f(x) = x^{\alpha}\,\chi_{(0,\,1)}$. Then $f \in L^2$. Set $f_n = (1 - 1/n)\,f$. Then $f_n \to f$ pointwise, but $||f - f_n||_r^r = ||(1/n)f||_r^r = \int_0^1 n^{-r}\,x^{\alpha r}\,dx = \infty$.

(b) Suppose $r \in [2, \infty)$. Then $h(x) = 3^r\,|f(x)|^r \leq 6^r\,|f(x)/2|^r \leq 6^r |f(x)/2|^2$, since $|f(x)/2| \leq 1$ and $r \geq 2$. We find $\int h \leq 6^r(1/4)||f||_2^2 < \infty$, so LDCT applies.

Suppose $r \in [1, 2)$. In this case, the LDCT does not apply, and we look for a counter-example. Pick a real number α such that $-\frac{1}{r} < \alpha < -\frac{1}{2}$, and set $f(x) = x^{\alpha}\,\chi_{[1,\,\infty)}$. Then $f \in L^2$. Set $f_n = (1 - 1/n)\,f$. Then $f_n \to f$ pointwise, but $||f - f_n||_r^r = ||(1/n)f||_r^r = \int_1^{\infty} n^{-r}\,x^{\alpha r}\,dx = \infty$.

(c) Suppose $r \in [2, 3]$. Then by interpolation (see Homework 14 – Problem 12.15), $f \in L^r$. It follows that $\int h < \infty$, and so the LDCT applies.

Suppose $r < 2$. In this case, construct a counter-example as in part (b) of a function f that does not decay fast enough to belong to L^r.

Suppose $r > 3$. In this case, construct a counter-example as in part (a) of a function f that has a sufficiently strong singularity that it does not belong to L^r.