Homework set 2 — APPM5450, Spring 2013

From the textbook: 7.13, 8.3, 8.4.

Problem 1: Let $T(t)$ denote the semigroup defined in Section 7.3 of the textbook. Prove that $T(t) \rightarrow I$ strongly as $t \searrow 0$. Prove that $T(t)$ does not converge in norm.

Problem 2: Prove that if P is a projection on a Hilbert space H, then the following three statements are equivalent:
(1) P is orthogonal, *i.e.* $\ker(P) = \operatorname{ran}(P)^{\perp}$.
(2) P is self-adjoint, *i.e.* $\langle P\,x,\,y\rangle = \langle x,\,P\,y\rangle \quad \forall x, y$.
(3) $||P|| = 0$ or 1.

Problem 3: This problem is just for fun (meaning that you can safely skip it if you're short on time). The complete solution is given in the first half of Section 7.5, but try to solve it without looking at the solution first.

The problem is to prove that if γ is a closed C^1 curve in the plane of length 2π, then the area enclosed by γ is less than or equal to π, with equality occurring if and only if γ is a circle.

We parameterize γ using curve-length s as the parameter. Let f and g be functions in $H^1(\mathbb{T})$ such that $\gamma(s) = [f(s),\,g(s)]$. Recall from Green's theorem that the area A enclosed by the curve is given by

$$A = \frac{1}{2}\int_\gamma \big(x\,dy - y\,dx\big) = \frac{1}{2}\int_0^{2\pi} \big(f(s)\,\dot{g}(s) - g(s)\,\dot{f}(s)\big)\,ds. \tag{1}$$

The problem is to find f and g that maximize A, subject to the constraint that the length or the curve is 2π:

$$2\pi = \int_0^{2\pi} \big(\dot{f}(s)^2 + \dot{g}(s)^2\big)\,ds. \tag{2}$$

Write f and g as Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n\,e^{inx}, \qquad g(x) = \sum_{n=-\infty}^{\infty} \beta_n\,e^{inx}.$$

Combine (1 and (2) to obtain

$$2\pi - 2A = \int_0^{2\pi} \big(\dot{f}(s)^2 + \dot{g}(s)^2 - f(s)\,\dot{g}(s) + g(s)\,\dot{f}(s)\big)\,ds. \tag{3}$$

Use Parseval's relation to rewrite (3) as a relation involving the Fourier coefficients α_n and β_n rather than f and g. Complete the squares to prove that $2\pi - 2A$ is non-negative (one good way of completing the squares will involve four squares, two of which are $|n\alpha_n - i\beta_n|^2$ and $|n\beta_n - i\alpha_n|^2$). Finally, prove that equality occurs if and only if γ is a circle.