Homework set 10 — APPM5450, Spring 2013 — Solution to 11.16 convolution

We will prove that for $\varphi \in \mathcal{S}(\mathbb{R})$ and $T = \mathcal{S}^*(\mathbb{R})$, we have

$$\mathcal{F}[\varphi * T] = \sqrt{2\pi}\,\hat{\varphi}\,\hat{T}.$$

First note that this part of the problem is *far* harder than the other ones.

The trickiness is due to the indirect definition $[\varphi * T](x) = \langle T,\, R\,\tau_x\varphi\rangle$. Note also that since $\varphi * T$ is not necessarily in L^1, we have to define $\mathcal{F}[\varphi * T]$ in a distributional sense. So fix $\psi \in \mathcal{S}$. Then

$$\begin{aligned}\langle \mathcal{F}[\varphi * T],\, \psi\rangle &\overset{(1)}{=} \langle \varphi * T,\, \hat{\psi}\rangle \overset{(2)}{=} \int_{\mathbb{R}} \langle T,\, R\,\tau_x\,\varphi\rangle\,\hat{\psi}(x)\,dx \overset{(3)}{=} \langle T,\, \int_{\mathbb{R}} \varphi(x-y)\,\hat{\psi}(x)\,dx\rangle \\ &\overset{(4)}{=} \langle T,\, (R\varphi) * \hat{\psi}\rangle \overset{(5)}{=} \langle T,\, \mathcal{F}\left(\mathcal{F}^*\left((R\varphi) * \hat{\psi}\right)\right)\rangle \overset{(6)}{=} \langle T,\, \mathcal{F}\left(\sqrt{2\pi}\,\hat{\varphi}\,\psi\right)\rangle \\ &\overset{(7)}{=} \langle \hat{T},\, \sqrt{2\pi}\,\hat{\varphi}\,\psi\rangle \overset{(8)}{=} \langle \sqrt{2\pi}\,\hat{\varphi}\,\hat{T},\, \psi\rangle\end{aligned}$$

Some comments on each step:

(1) Simply the definition of the distributional Fourier transform.

(2) Use that $\varphi * T$ is a "plain" tempered function, and $\hat{\psi} \in \mathcal{S}(\mathbb{R})$.

(3) Here is *des Pudels Kern.* We need to move the integral inside the $\mathcal{S}^* \times \mathcal{S}$ pairing. First observe that the integrand is smooth and rapidly decaying, so Riemann sums converge nicely. Consider a Riemann sum on the interval $[-n, n]$, with spacing $1/n$. Then

$$\begin{aligned}\int_{\mathbb{R}} \langle T,\, R\,\tau_x\,\varphi\rangle\,\hat{\psi}(x)\,dx &= \lim_{n\to\infty} \sum_{j=-n^2}^{n^2} \frac{1}{n}\langle T,\, R\,\tau_{j/n}\,\varphi\rangle\,\hat{\psi}(j/n) \\ &= \lim_{n\to\infty} \langle T,\, \sum_{j=-n^2}^{n^2} \frac{1}{n}\varphi(j/n - y)\,\hat{\psi}(j/n)\rangle.\end{aligned}$$

In order to justify taking the limit inside the $\mathcal{S}^* \times \mathcal{S}$ pairing, we invoke the continuity of T, and then "only" need to prove that

$$\sum_{j=-n^2}^{n^2} \frac{1}{n}\varphi(j/n - \cdot)\,\hat{\psi}(j/n) \mapsto \int_{\mathbb{R}} \varphi(x - \cdot)\,\psi(x)\,dx, \qquad \text{as } n \to \infty,$$

where the convergence is in $\mathcal{S}$. We leave the details as an exercise. ⌣̈

(4) Observe that the function $y \mapsto \int \varphi(x-y)\hat{\psi}(x)dx$ is the convolution between $R\varphi$ and $\hat{\psi}$.

(5) Use that $\mathcal{F}\,\mathcal{F}^* = I$ on $\mathcal{S}$.

(6) Use that $\mathcal{F}^*(f * g) = \sqrt{2\pi}\,(\mathcal{F}^* f)\,(\mathcal{F}^* g)$ for any $f, g \in \mathcal{S}$, and that $\mathcal{F}^*\,R = \mathcal{F}$.

(7) Simply the definition of the distributional Fourier transform.

(8) Definition of multiplication by a Schwartz function.