Homework set 12 — APPM5450, Spring 2013 — Solutions

Problem 12.2: We use "$\uplus$" to denote disjoint unions.

(a) Suppose that $A, B \in \mathcal{A}$. Then note that $A\backslash B = A \cap B^{\rm c} = (A^{\rm c} \cup B)^{\rm c}$. It now follows directly from the axioms that $A\backslash B \in \mathcal{A}$.

(b) Set $C = B\backslash A$. Then $B = A \uplus C$, so

$$\mu(B) = \mu(A \uplus C) = \mu(A) + \mu(C) \geq \mu(A).$$

(c) Set $C = A \cap B$. Then $A = (A\backslash B) \uplus C$ and $B = (B\backslash A) \uplus C$ so

$$\begin{aligned}\mu(A \cup B) = \mu((A\backslash B) \uplus C \uplus (B\backslash A)) &= \mu(A\backslash B) + \mu(C) + \mu(B\backslash A)\\ &\leq \mu(A\backslash B) + \mu(C) + \mu(C) + \mu(B\backslash A)\\ &= \mu((A\backslash B) \uplus C) + \mu(C \uplus (B\backslash A)) = \mu(A) + \mu(B).\end{aligned}$$

Problem 12.3: The trick is to write $\bigcup_{n=1}^{\infty} A_n$ as a disjoint union. For $n = 1, 2, 3, \dots$ set $B_n = A_{n+1}\backslash A_n$. Then

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup \left(\biguplus_{n=1}^{\infty} B_n\right),$$

where there union on the right is a disjoint one. Now use additivity twice:

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(A_1 \cup \left(\biguplus_{n=1}^{\infty} B_n\right)\right) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(B_n)\\ &= \lim_{N\to\infty}\left(\mu(A_1) + \sum_{n=1}^{N} \mu(B_n)\right) = \lim_{N\to\infty} \mu\left(A_1 \cup \left(\biguplus_{n=1}^{N} B_n\right)\right) = \lim_{N\to\infty} \mu\left(A_N\right).\end{aligned}$$

For the second part, set $C = \cap_{n=1}^{\infty} A_n$ and $C_n = A_n\backslash A_{n+1}$. Then

$$\mu(A_N) = \mu\left(C \cup \left(\biguplus_{n=N}^{\infty} C_n\right)\right) = \mu(C) + \sum_{n=N}^{\infty} \mu(C_n).$$

Since $\infty > \mu(A_1) \geq \sum_{n=1}^{\infty} \mu(C_n)$, we find that

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(C_n) = 0,$$

which completes the proof. For the counterexample, consider $X = \mathbb{R}^2$, and $A_n = \{x = (x_1,\, x_2) :\, |x_2| < 1/n\}$. Then $\mu(A_n) = \infty$ for all n, but $\cap_{n=1}^{\infty} A_n$ is the x_1-axis, which has measure zero.

Problem 12.5: Straight-forward.

Problem 12.7:

Reflexivity: It is obvious that $f(x) = f(x)$ a.e.

Symmetry: If $f(x) = g(x)$ a.e., then obviously $g(x) = f(x)$ a.e.

Transitivity: Suppose that $f(x) = g(x)$ a.e. and that $g(x) = h(x)$ a.e. Set

$$A = \{x : \ f(x) \neq g(x)\}$$
$$B = \{x : \ g(x) \neq h(x)\}$$
$$C = \{x : \ f(x) \neq h(x)\}.$$

We know that $\mu(A) = \mu(B) = 0$, and we want to prove that $\mu(C) = 0$. It is clearly the case that $C \subseteq A \cup B$, and then it follows directly that $\mu(C) \leq \mu(A) + \mu(B) = 0$.