APPM5450 — Applied Analysis: Section exam 1
8:30 – 9:50, February 25, 2013. Closed books.

The following problems are worth 25 points each.

Problem 1: Let H be a Hilbert space with an ON-basis $(e_j)_{j=1}^{\infty}$.

(a) State what it means for a sequence $(u_n)_{n=1}^{\infty}$ in H to *converge weakly* to a vector $u \in H$.

(b) Suppose that you are given a sequence of vectors $(u_n)_{n=1}^{\infty}$ for which you know:
(1) There exists a finite M such that $||u_n|| \leq M$ for every n.
(2) There is a vector $u \in H$ such that for every j, we have $\lim_{n\to\infty} (e_j, u_n) = (e_j, u)$.
Is is necessarily the case that (u_n) converges weakly to u? Either prove directly from the definition you gave in (a) that this is true, or give a counter-example.

Problem 2: Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a self-adjoint operator. Let b be a non-zero real number. Prove that the operator $B = A + i\,b\,I$ has closed range (where "i" is the imaginary unit). Is B necessarily one-to-one? Is B necessarily onto?

Problem 3: Set $I = [-\pi,\, \pi]$, and consider for $n = 1,\, 2,\, 3,\, \dots$ the functions

$$u_n(x) = \sum_{j=1}^{n} \frac{1}{j^{7/4}} \cos\big((2j+1)x\big).$$

(a) Does the sequence $(u_n)_{n=1}^{\infty}$ converge in $L^2(I)$? In $C(I)$? In $C^1(I)$? In $H^k(I)$ for any positive k? Please motivate your answers briefly.

(b) What can you tell about the sequence $(u_n')_{n=1}^{\infty}$ of *derivatives* of u_n's? Does it converge in any of the spaces mentioned in part (a)?

Problem 4: Set $I = [-\pi,\, \pi]$, $H = L^2(I)$, and let $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ denote the standard Fourier basis for H. Define for $t \geq 0$, the operator $A(t)$ via

$$A(t)u = \sum_{n=-\infty}^{\infty} e^{-n^2 t}\,(e_n, u)\, e_n.$$

(a) Prove that for any $t \geq 0$, the operator $A(t)$ is continuous and determine $||A(t)||$.

(b) Prove that for any $t > 0$ and for any $u \in H$, it is the case that $A(t)\,u \in C^k(I)$ for any k.

(c) Fix $u \in H$ and define for $t > 0$ the function $v(t,x) = A(t)u$. State a second order partial differential equation that v satisfies (with boundary conditions). No motivation required.

(d) Define for $m = 1,\, 2,\, 3,\, \dots$ the operator $B_m = A(1/m)$. Does $(B_m)_{m=1}^{\infty}$ converge in $\mathcal{B}(H)$? If so, to what? In what sense? No motivation required.