APPM5450 — Applied Analysis: Section exam 2 — Solutions

8:30 – 9:50, March 20, 2013. Closed books.

Problem 1: (5p) Let A be a compact self-adjoint operator on a Hilbert space H. You know that $\sigma(A) = \{0\} \cup \{1,\, 1/2,\, 1/3,\, 1/4,\, \dots\}$. Set $\lambda = 1/\pi$. What is $||(A - \lambda I)^{-1}||$? No motivation required.

Solution: Recall that $||(A - \lambda I)^{-1}|| = 1/d$ where $d = \text{dist}(\lambda, \sigma(A)) = \inf\{|\lambda - \mu| :\ \mu \in \sigma(A)\}$.

In this case, $d = \frac{1}{3} - \frac{1}{\pi}$ so $||(A - \lambda I)^{-1}|| = \frac{1}{\frac{1}{3} - \frac{1}{\pi}} = \frac{3\pi}{\pi - 3}$.

Problem 2: (5p) Let H be a Hilbert space, and suppose that $A \in \mathcal{B}(H)$. You know that $||A|| = 2$ and $A^3 = 0$. What can you say about $\sigma(A)$? Please motivate briefly.

Solution: Recall that $r(A) = \lim_{n\to\infty} ||A^n||^{1/n}$. In this case, $A^n = 0$ when $n \geq 3$, so $r(A) = 0$. It follows that $\sigma(A) = \{0\}$.

Partial credit is given for the answer $\sigma(A) \subseteq \{\lambda \in \mathbb{C} :\ |\lambda| \leq 2\}$. (Note however that since A is not necessarily self-adjoint, you cannot state that $\sigma(A) \subseteq [-2, 2]$.)

Problem 3: (10p) Set $H = L^2(\mathbb{R})$, and $[Au](x) = u(x - 1)$. No motivation required.

(a) (4p) Specify $\inf_{\lambda \in \sigma(A)} |\lambda|$. $= 1$

(b) (4p) Specify $\sup_{\lambda \in \sigma(A)} |\lambda|$. $= 1$

(c) (2p) Specify $\sigma(A)$. $= \{\lambda \in \mathbb{C} :\ |\lambda| = 1\}$

Solution: The operator A is unitary since $[A^* u](x) = u(x + 1)$ and so $A^* = A^{-1}$. It then follows that $\sigma(A) \subseteq \{\lambda \in \mathbb{C} :\ |\lambda| = 1\}$. The answers to (a) and (b) follow immediately from this statement. Note that even if you did not observe that A is unitary, you can easily get the correct answer to (b) by simply observing that $||A|| = 1$.

To get the answer to (c), proceed as follows: Suppose $\lambda = e^{i\theta}$ for some real number $\theta \in [-\pi,\, \pi]$. Then the function $U(x) = e^{-i\theta x}$ is *almost* an eigenvector of A. The problem is that $||U||_{L^2} = \infty$. But now set $u_N = (2N)^{-1/2}\, U\, \chi_{[-N,N]}$, observe that $||u_N||_{L^2} = 1$, and $\lim_{N\to\infty} ||(A - e^{i\theta})\, u_N|| = 0$ to establish that $e^{i\theta} \in \sigma(A)$.

Problem 4: (30p) Consider the Hilbert space $H = L^2(\mathbb{R})$. Define a function $f \in C_{\rm b}(\mathbb{R})$ via

$$f(x) = \begin{cases} \arctan(x) & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Define a linear operator $A:\ H \rightarrow H$ via

$$[Au](x) = f(x)\,u(x).$$

(a) (5p) Specify $||A||$. No motivation required.

(b) (5p) Is A self-adjoint? Compact? Non-negative? Positive? No motivation required.

(c) (12p) Specify $\sigma(A)$, $\sigma_{\rm r}(A)$, $\sigma_{\rm c}(A)$, and $\sigma_{\rm p}(A)$. No motivation required.

(d) (8p) Provide brief motivations for your claims in (c).

Solution:

(a) $\pi/2$

(b) A is S-A and non-negative; not compact; not positive.

(c)

$\sigma_{\rm r}(A) = \emptyset$ since A is S-A.

$\sigma_{\rm p}(A) = \{0\}$. First note that $Au = 0$ if u is supported in $(-\infty, 0]$, this shows that $0 \in \sigma_{\rm p}(A)$. Next suppose $\lambda \neq 0$. Then show that if $(f(x) - \lambda)\,u(x) = 0$, then $u(x) = 0$ except possibly for at a single point. This shows that $A - \lambda I$ is one-to-one, and so $0 \notin \sigma_{\rm p}(A)$.

$\sigma_{\rm c}(A) = (0, \pi/2]$. Since A is non-negative and $||A|| = \pi/2$, we know that $\sigma_{\rm c}(A) \subseteq [0, \pi/2]$. Since $\sigma_{\rm p}(A) = \{0\}$, we can further conclude that $\sigma_{\rm c}(A) \subseteq (0, \pi/2]$. We have already shown that for $\lambda \in (0, \pi/2]$, the operator $A - \lambda I$ is one-to-one. To complete the argument, simply prove that $A - \lambda I$ is not coercive.

$\sigma(A) = [0, \pi/2]$

Problem 5: (25p) Consider four maps $T_i : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ defined via

$$
\begin{aligned}
T_1(\varphi) &= \sup_{x\in\mathbb{R}} \varphi(x) \\
T_2(\varphi) &= \varphi'(2) \\
T_3(\varphi) &= \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{1}{x}\varphi(x)\,dx \\
T_4(\varphi) &= \lim_{\varepsilon\to 0} \left[\int_{|x|\geq\varepsilon} \frac{1}{x^2}\varphi(x)\,dx - \frac{2\varphi(0)}{\varepsilon} \right].
\end{aligned}
$$

No motivation required for these problems:

(a) (11p) Which of the maps T_i belong to $\mathcal{S}^*(\mathbb{R})$?

(b) (7p) For the maps that do belong to $\mathcal{S}^*(\mathbb{R})$, state the *order* of the map.

(c) (7p) For the maps that do belong to $\mathcal{S}^*(\mathbb{R})$, specify a continuous function f_i on $\mathbb{R}$ such that T_i is a derivative (first, second, third, ...) of f_i.

Solution:

- T_1 is not linear.
- T_2 has degree 1. (In fact $T_2 = -\delta_2'$). Set $f(x) = -\frac{1}{2}|x-2|$, then $T_2 = \partial^3 f$.
- T_3 has degree 1 (see course notes). Set $g(x) = x\log|x| - x$. Then $T_3 = \partial^2 g$.
- T_4 is the derivative of T_3 and has degree 2. So $T_4 = \partial^3 g$, with $g(x) = x\log|x| - x$.

Problem 6: (25p) Let φ be a Schwartz function on $\mathbb{R}$. Define for $N = 1, 2, 3, \ldots$, the function ψ_N via

$$\psi_N(x) = \sum_{n=1}^{N} \frac{1}{n^2} \varphi(x-n).$$

Please motivate your answers to the following questions *briefly.*

(a) (7p) Is the function ψ_N necessarily a Schwartz function?

(b) (9p) Does the sequence $(\psi_N)_{N=1}^{\infty}$ necessarily converge in $C_{\mathrm{b}}(\mathbb{R})$?

(c) (9p) Does the sequence $(\psi_N)_{N=1}^{\infty}$ necessarily converge in $\mathcal{S}(\mathbb{R})$?

Solution:

(a) Yes, ψ_N is a finite sum of Schwartz functions and since $\mathcal{S}$ is a linear space, $\psi_N \in \mathcal{S}$. (Observe that if $\varphi \in \mathcal{S}$, then $\tau_n \varphi \in \mathcal{S}$.)

(b) Yes, each $\psi_N \in C_{\mathrm{b}}$, and we can prove that the sequence is Cauchy w.r.t. the uniform norm: Given N, pick m and n such that $N \leq m \leq n$, then

$$||\psi_n - \psi_m||_{\mathrm{u}} = ||\sum_{j=m+1}^{n} \frac{1}{j^2} \tau_j \varphi||_{\mathrm{u}} \leq \sum_{j=m+1}^{n} \frac{1}{j^2} ||\tau_j \varphi||_{\mathrm{u}} = ||\varphi||_{\mathrm{u}} \sum_{j=m+1}^{n} \frac{1}{j^2} \leq ||\varphi||_{\mathrm{u}} \sum_{j=N+1}^{\infty} \frac{1}{j^2} \to 0$$

as $N \to \infty$.

(c) The sequence typically does *not* converge in $\mathcal{S}$. For instance, set $\varphi(x) = e^{-x^2}$. Then the uniform limit of ψ_N is the function

$$\psi(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-(x-n)^2}$$

which does not belong to $\mathcal{S}$ since it does not decay faster than any polynomial. In particular,

$$||\psi||_{0,3} = \sup_{x \in \mathbb{R}} (1 + |x|^6)^{1/2} \, |\psi(x)| \geq \sup_{n \in \mathbb{N}} (1 + |n|^6)^{1/2} \, |\psi(n)| \geq \sup_{n \in \mathbb{N}} (1 + |n|^6)^{1/2} \, \frac{1}{n^2} = \infty.$$