APPM5450 — Applied Analysis: Section exam 3
8:30 – 9:50, April 24, 2013. Closed books.

Problem 1: (36p) No motivation required.

(a) (6p) Define $T \in \mathcal{S}^*(\mathbb{R}^d)$ via $\langle T, \varphi\rangle = \varphi(0)$. What is $\hat{T}$?

(b) (6p) Define $T \in \mathcal{S}^*(\mathbb{R}^d)$ via $\langle T, \varphi\rangle = \varphi(0)$. State for which $s \in \mathbb{R}$ (if any) we have

$$\int_{\mathbb{R}^d} \big(1 + |t|^2\big)^s\, |\hat{T}(t)|^2\, dt < \infty.$$

(Recall that this is the definition for when $T \in H^s(\mathbb{R}^d)$).

(c) (6p) Define $T \in \mathcal{S}^*(\mathbb{R})$ via $T(x) = \chi_{[0,1]}(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$. What is $\hat{T}$?

(d) (6p) Define $T \in \mathcal{S}^*(\mathbb{R})$ via $T(x) = \text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$. What is $\hat{T}$?

(e) (6p) Give an example of a non-zero Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}\varphi = -i\,\varphi$, where i is the imaginary unit.

(f) (6p) Which of the following statements are true:
(1) $\mathcal{F}(L^2(\mathbb{R})) \subseteq C_0(\mathbb{R})$
(2) $\mathcal{S}(\mathbb{R})$ is dense in $\mathcal{S}^*(\mathbb{R})$
(3) $L^p(\mathbb{R}) \subseteq \mathcal{S}^*(\mathbb{R})$ for every $p \in [1,\infty]$.
(4) $H^s(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$ when $s > d/2$.
(5) $C^1(\mathbb{R}) \subseteq H^2(\mathbb{R})$.

Solution:

(a) $\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle = \hat{\varphi}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\,0\,x}\varphi(x)\,dx$ so $\hat{T} = 1/\sqrt{2\pi}$.

(b) $\int_{\mathbb{R}^d}(1+|t|^2)^s\,|\hat{T}(t)|^2\,dt \sim \int_0^\infty (1+r^2)^s\,r^{d-1}\,dr$ which is finite iff $2s + d - 1 < -1$, which is to say, if $s < -d/2$.

(c) Note that $T \in L^1$ so $\hat{T}(t) = \frac{1}{\sqrt{2\pi}}\int_0^1 e^{-ixt}\,dx = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{-it}e^{-ixt}\right]_0^1 = \frac{1}{\sqrt{2\pi}\,i\,t}\big(1 - e^{-ix}\big)$.

(d) $\hat{T} = \frac{-2i}{\sqrt{\pi}}\text{P.V.}\left(\frac{1}{t}\right)$, see Problem 11.22 on Homework 11.

To get a clue of what the solution might be, note that $T' = 2\delta$. Then $it\,\hat{T}(t) = 2/\sqrt{2\pi}$ which would indicate $\hat{T}(t) \sim i/t$.

(e) For instance, $\varphi(x) = x\,e^{-x^2/2}$. (See material on eigen-vectors of $\mathcal{F}$ in lecture notes.) Note that in the exam actually administered, the problem stated was to find φ such that $\mathcal{F}\varphi = i\varphi$. This is harder since the simplest solution is a third degree polynomial times $e^{-x^2/2}$. In consequence, this problem was graded very generously. Any hint towards Hermite functions gave almost full points.

(f) The answers are:

(1) FALSE — note that $\mathcal{F}(L^2) = L^2$ and L^2-functions are not necessarily continuous.
(2) TRUE.
(3) TRUE — recall that any L^p function is "tempered" by virtue of the Hölder inequality.
(4) TRUE — this is the simplest form of the Sobolev embedding theorem.
(5) FALSE — for instance, consider $f(x) = 1$, then $f \in C^1$ but $f \notin H^2$.

Problem 2: (24p) Let R denote a real number such that $0 < R < \infty$ and define

$$f_n(x) = \begin{cases} n\,\cos(nx) & \text{for } |x| \leq R, \\ 0, & \text{for } |x| > R. \end{cases}$$

For which numbers R, if any, is it the case that $f_n \to 0$ in $\mathcal{S}^*(\mathbb{R})$? Please motivate your answer.

Solution: Fix $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned} \langle f_n,\, \varphi\rangle &= \int_{-R}^{R} n\,\cos(nx)\,\varphi(x)\,dx = [\sin(nx)\,\varphi(x)]_{-R}^{R} - \int_{-R}^{R} \sin(nx)\,\varphi'(x)\,dx \\ &= \underbrace{[\sin(nx)\,\varphi(x)]_{-R}^{R}}_{=:I_1^{(n)}} + \underbrace{\left[\frac{\cos(nx)}{n}\,\varphi'(x)\right]_{-R}^{R}}_{=:I_2^{(n)}} - \underbrace{\int_{-R}^{R} \frac{\cos(nx)}{n}\,\varphi''(x)\,dx}_{=:I_3^{(n)}} \end{aligned}$$

We see that for any R, we have

$$|I_2^{(n)}| \leq \frac{|\varphi'(-R)| + |\varphi'(R)|}{n} \leq \frac{2||\varphi'||_{\mathrm{u}}}{n} \leq \frac{2||\varphi||_{1,0}}{n} \to 0 \qquad \text{as } n \to \infty,$$

and also

$$|I_3^{(n)}| \leq \frac{1}{n}\int_{-R}^{R} |\varphi''(x)|\,dx \leq \frac{1}{n}\int_{-\infty}^{\infty} (1+x^2)^{-1}\,(1+x^2)|\varphi''(x)|\,dx \leq \frac{1}{n}\,\pi\,||\varphi||_{2,2} \to 0 \qquad \text{as } n \to \infty.$$

So the question reduces to whether $I_1^{(n)} \to 0$ as $n \to \infty$. We see that

$$I_1^{(n)} = \sin(nR)\,\big(\varphi(R) + \varphi(-R)\big).$$

Case 1: If R is a multiple of π, then $I_1^{(n)} = 0$ for every n, and so $f_n \to 0$ in $\mathcal{S}^*$.

Case 2: If R is not a multiple of π, then $I_1^{(n)}$ will not converge for any φ such that $\varphi(R) + \varphi(-R) \neq 0$, so in this case (f_n) is not convergent.

Problem 3: (20p)

(a) (5p) State the definition of a σ-algebra.

(b) (5p) State the definition of a measure.

(c) (10p) Let $(X, \mathcal{A}, \mu)$ denote a measure space. Suppose that $\Omega_1,\ \Omega_2 \in \mathcal{A}$. Prove directly from the axioms that $\mu(\Omega_1 \cup \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$. Give a condition for when equality occurs.

Solution: Set

$$\begin{aligned}\Psi_1 =& \Omega_1 \backslash \Omega_2 = \Omega_1 \cap \Omega_2^{\mathrm{c}},\\ \Psi_2 =& \Omega_2 \backslash \Omega_1 = \Omega_2 \cap \Omega_1^{\mathrm{c}},\\ \Psi_3 =& \Omega_1 \cap \Omega_2.\end{aligned}$$

We see from the axioms of a σ-algebra that $\Psi_1,\ \Psi_2,\ \Psi_3 \in \mathcal{A}$.

Now observe that

$$\mu(\Omega_1 \cup \Omega_2) \overset{(1)}{=} \mu(\Psi_1 \cup \Psi_2 \cup \Psi_3) \overset{(2)}{=} \mu(\Psi_1) + \mu(\Psi_2) + \mu(\Psi_3) \overset{(3)}{\leq} \mu(\Psi_1) + 2\mu(\Psi_2) + \mu(\Psi_3) \overset{(4)}{=} \mu(\Omega_1) + \mu(\Omega_2).$$

The steps are justified as follows:

(1) Since $\Omega_1 \cup \Omega_2 = \Psi_1 \cup \Psi_2 \cup \Psi_3$.

(2) The sets $\Psi_1,\ \Psi_2,\ \Psi_3$ are mutually disjoint, so the axiom for additivity of the measure applies.

(3) Since $\mu(\Psi_2) \geq 0$ by an axiom for measures.

(4) Since $\Omega_1 = \Psi_1 \cup \Psi_2$, since $\Omega_2 = \Psi_2 \cup \Psi_3$, and since $\Psi_1 \cap \Psi_2 = \emptyset$ and $\Psi_2 \cap \Psi_3 = \emptyset$.

We see that the inequality (3) is a strict inequality unless $\mu(\Psi_2)$ is zero. Consequently:

$$\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2) \qquad \Leftrightarrow \qquad \mu(\Omega_1 \cap \Omega_2) = 0.$$

Problem 4: (20p) Set $X = [1, \infty)$, let $\mathcal{A}$ denote the power set on X, let $\mathbb{N} = \{0,\, 1,\, 2,\, 3,\, \dots\}$ denote the natural numbers, and define a measure μ on $\mathcal{A}$ via

$$\mu(\Omega) = \sum_{j \in \Omega \cap \mathbb{N}} 2^{-j}.$$

Which of the following functions are Lebesgue integrable with respect to $(X, \mathcal{A}, \mu)$? For the functions that are Lebesgue integrable, state the value of $\int_X f\, d\mu$. Please motivate briefly.

(a) $f_1(x) = e^x$

(b) $f_2(x) = e^{-x}$

(c) $f_3(x) = e^x - 9\, e^{-x}$

(d) $f_4(x) = e^x\, \cos(\pi x)$

Solution: Since $\mathcal{A}$ is the power set, *every* function on this measure space is measurable.

(a) f_1 is non-negative so it is certainly integrable. We find

$$\int_X f_1\, d\mu = \sum_{j=1}^{\infty} e^j 2^{-j} = \sum_{j=1}^{\infty} (e/2)^j = \infty$$

since $e/2 \geq 1$.

(b) f_2 is non-negative so it is certainly integrable. We find

$$\int_X f_2\, d\mu = \sum_{j=1}^{\infty} e^{-j} 2^{-j} = \sum_{j=1}^{\infty} (1/2e)^j = \frac{1/2e}{1 - 1/2e} = \frac{1}{2e-1}.$$

(c) Set $t = \log(3)$. Then $f_3 = f_+ - f_-$ where $f_- = f_3\, \chi_{[0,t]}$ and $f_+ = f_3\, \chi_{[t,\infty)}$. We find that

$$\int_x f_-\, d\mu = (1/2)\,(e - 9\, e^{-1})$$

and that

$$\int_x f_+\, d\mu = \sum_{j=2}^{\infty} 2^{-j}\left(e^j - 9\, e^{-j}\right) = \infty.$$

Since only one of the two integrals is infinite, we find that f_3 is integrable, and

$$\int_x f_3\, d\mu = \infty - (1/2)\,(e - 9\, e^{-1}) = \infty.$$

(d) Decompose $f_4 = f_+ - f_-$ with $f_\pm \geq 0$ as usual. We find that

$$\int_x f_+\, d\mu = \sum_{j=2,\,4,\,6,\,8,\,\dots} 2^{-j}\, e^j = \infty,$$

and also

$$\int_x f_-\, d\mu = \sum_{j=1,\,3,\,5,\,7,\,\dots} 2^{-j}\, e^j = \infty.$$

Since both f_+ and f_- have infinite integrals, f_4 is *not* Lebesgue integrable.