9.1) Prove that \( \rho(A^*) = \overline{\rho(A)} \) where \( \overline{\rho(A)} \) is the set \( \{ \lambda \in \mathbb{C} | \overline{\lambda} \in \rho(A) \} \).

Assume \( \lambda \in \rho(A) \). Then \( (A - \lambda I)^{-1} \) and \( (A - \lambda I)^{-1} \) exist and are bounded. We need to show that \( ((A - \lambda I)^{-1})^* = (A^* - \overline{\lambda} I)^{-1} \).

If we show that \( (B^{-1})^* = (B^*)^{-1} \) then this follows immediately.

Say \( y = By' \). Then \( \langle (B^*)^{-1} x, y \rangle = \langle (B^*)^{-1} x, By' \rangle = \langle B' (B^*)^{-1} x, y' \rangle = \langle x, y' \rangle \)

So \( (B^{-1})^* = (B^*)^{-1} \) and we are done.

9.2) If \( \lambda \) is an eigenvalue of \( A \) then \( \overline{\lambda} \) is in the spectrum of \( A^* \). What can you say about the type of spectrum \( \overline{\lambda} \) belongs to?

First we show that \( \overline{\lambda} \) is in the spectrum of \( A^* \): \( \lambda \in \sigma_p(A) \Rightarrow \exists x \neq 0 \text{ s.t. } (A - \lambda I) x = 0 \forall y \)

This holds iff \( \langle x, (A^* - \overline{\lambda} I)y \rangle = 0 \forall y \) which holds iff \( x \perp ran(A^* - \overline{\lambda} I) \Rightarrow \overline{\lambda} \in \sigma(A^*) \)

Now \( \overline{\lambda} \not\in \sigma_c(A) \) because \( (A^* - \overline{\lambda} I) \) is dense iff \( (A^* - \overline{\lambda} I)^{\perp} = 0 \), but \( x \neq 0 \).

So \( \overline{\lambda} \) is in either the point or residual spectrum of \( A^* \).
9.3) Suppose that \( A \) is a bounded linear operator of a Hilbert space and \( \lambda, \mu \in \rho(A) \). Prove that the resolvent \( R_\lambda \) of \( A \) satisfies \( R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu \).

First note that \( A^{-1} - B^{-1} = -A^{-1}(A - B)B^{-1} \) (we use this below in the equality denoted by *).

Then \( R_\lambda - R_\mu = (A - \lambda I)^{-1} - (A - \mu I)^{-1} = -\frac{(A - \lambda I)^{-1}((A - \lambda I) - (A - \mu I)(A - \mu I)^{-1})}{(A - \lambda I)} = (\mu - \lambda)R_\lambda R_\mu \).

9.4) Prove that the spectrum of an orthogonal projection \( P \) is either \( \{0\} \) (in which case \( P = 0 \)), \( \{1\} \) (in which case \( P = I \)), or \( \{0,1\} \).

Assume that \( P \) is an orthogonal projection. Then \( H = \text{ran}P \oplus \ker P \) where \( \text{ran}P = (\ker P)^\perp \).

**Case 1**) \( \text{ran}P = \{0\} \)

Then \( P = 0 \) and \( Px = 0 \ \forall x \) so \( 0 \in \sigma_p(P) \)

If \( \lambda \neq 0 \) then \( (P - \lambda I)^{-1} = \frac{1}{\lambda}I \) so \( \lambda \in \rho(P) \)

**Case 2**) \( \ker P = \{0\} \)

Then \( \text{ran}P = (\ker P)^\perp = H \) so \( P = I \) and \( Px = x \ \forall x \) so \( 1 \in \sigma_p(P) \)

If \( \lambda \neq 1 \) then \( (P - \lambda I)^{-1} = \frac{1}{1-\lambda}I \) so \( \lambda \in \rho(P) \)

**Case 3**) \( \text{ran}P \neq \{0\}, \ker P \neq \{0\} \)

If \( x \neq 0, x \in \text{ran}P \) then \( x = Px \) so \( 1 \in \sigma_p(P) \)

If \( x \neq 0, x \in \ker P \) then \( 0 = Px \) so \( 0 \in \sigma_p(P) \)

If \( \lambda \neq 0,1 \) then \( (P - \lambda I)^{-1} = \frac{1}{1-\lambda}P - \frac{1}{\lambda}(I - P) \) so \( \lambda \in \rho(P) \)
9.5) \( A \) is a bounded, nonnegative operator on a complex Hilbert space. Prove that \( \sigma(A) \subset [0, \infty) \).

First note that \( A \) nonnegative implies \( A \) self-adjoint and \( A \) self-adjoint implies \( \sigma(A) \in \mathbb{R} \). Also, \( A \) bounded implies \( \sigma(A) \subseteq [-\|A\|, \|A\|] \).

Assume \( \lambda < 0 \). We need to show that \( (A - \lambda I) \) is invertible. Since \( A \) is self-adjoint we know that \( (Au, u) = (u, Au) \in \mathbb{R} \) so:

\[
\| (A - \lambda I) u \|^2 = \|Au\|^2 - 2\lambda (Au, u) + \lambda^2 \|u\|^2 \geq \lambda^2 \|u\|^2
\]

so \( (A - \lambda I) \) is coercive. A coercive implies

\[
\begin{cases}
\text{ran}(A - \lambda I) \text{ closed} \Rightarrow \lambda \notin \sigma_c(A) \\
\text{(A - \lambda I) one-to-one} \Rightarrow \lambda \notin \sigma_p(A)
\end{cases}
\]

A self-adjoint implies \( \sigma_r(A) = \{ \text{empty} \} \).

Since \( \lambda \) is not in any of the parts of the spectrum it is not in the spectrum and our proof is complete.
9.6) G is a multiplication operator on $L^2(R)$ defined by $Gf(x) = g(x)f(x)$ where $g$ is continuous and bounded. Prove that $G$ is a bounded linear operator on $L^2(R)$ and that its spectrum is given by $\sigma(G) = \{g(x) | x \in R\}$. Can an operator of this form have eigenvalues?

**G is a bounded linear operator:**

$$\|G\| = \sup_{\|f\|=1} \|Gf\| = \sup_{\|f\|=1} \left( \int |g(x)f(x)|^2 \, dx \right)^{1/2} \leq \sup_{\|f\|=1} \left( \int |f(x)|^2 \, dx \right)^{1/2} \leq \|g\|_\infty$$

**Spectrum:** Set $\Omega = \{g(x) | x \in R\}$.

Suppose $\lambda \not\in \Omega$. Then $\exists \varepsilon > 0$ s.t. $|\lambda - g(x)| \geq \varepsilon \ \forall x$.

Note that

$$(G - \lambda I) \frac{1}{g(x) - \lambda} f(x) = \frac{g(x)}{g(x) - \lambda} f(x) - \frac{\lambda}{g(x) - \lambda} f(x) = f(x) \Rightarrow (G - \lambda I)^{-1} f(x) = \frac{1}{g(x) - \lambda} f(x)$$

Then

$$\|(G - \lambda I)^{-1}\| = \sup_{\|f\|=1} \left( \int \left| \frac{1}{g(x) - \lambda} f(x) \right|^2 \, dx \right)^{1/2} \leq \sup_{\|f\|=1} \frac{1}{\|g(x) - \lambda\|} \sup_{\|f\|=1} \left( \int |f(x)|^2 \, dx \right)^{1/2} \leq \varepsilon$$

Suppose $\lambda \in \Omega$. Then there exists $x_n \in R$ s.t. $g(x_n) \to \lambda$.

For $j = 1, 2, 3...$ pick $n_j$ s.t. $|g(x_{n_j}) - \lambda| \leq \frac{1}{j}$

Since $g$ is continuous at $x_{n_j}$, there exists $\delta$ s.t. $x \in B_\delta(x_{n_j}) \Rightarrow |g(x) - g(x_n)| \leq \frac{1}{j}$

Set $u_{n_j}(x) = \begin{cases} \sqrt{j/2} & x \in B_\delta(x_{n_j}) \\ 0 & \text{else} \end{cases}$

Then

$$\left\| (G - \lambda I) u_{n_j} \right\|^2 = \int |g(x) - \lambda|^2 u_{n_j}(x)^2 \, dx \leq \int \left[ |g(x) - g(x_{n_j})| + |g(x_{n_j}) - \lambda| \right]^2 u_{n_j}(x)^2 \, dx \leq j \int |u_{n_j}(x)|^2 \, dx \leq \frac{4}{j^2} \int |u_{n_j}(x)|^2 \, dx = \frac{4}{j^2} \to 0$$

The inequality denoted by “TI” uses the triangle inequality.

We have shown that $(G - \lambda I)$ is not continuously invertible (so $\lambda$ is in the spectrum).

**Eigenvalues:** Suppose $(G - \lambda I)u = 0$ for $u \neq 0$.

Then $(g(x) - \lambda)u(x) = 0$ but $u \neq 0$. This is possible if and only if the set $\{x : g(x) = \lambda\}$ has positive (non-zero) measure.
9.7) Let \( K: L^2([0,1]) \rightarrow L^2([0,1]) \) be the integral operator defined by \( Kf(x) = \int_0^x f(y)dy \).

(a) Find the adjoint operator \( K^* \).

\[
(Kf, g) = \int_0^1 \int_0^1 f(y)g(x)dxdy = \int_0^1 \int_0^1 f(y)g(x)dydx = \int_0^1 f(y)g(x)dxdy = \int_0^1 f(y)g(x)dydx = (f, K^*g)
\]

So \( K^*g(x) = \int_y^1 g(y)dy \)

(b) Show that \( \|K\| = 2/\pi \).

Set \( \phi_n(x) = \sqrt{2} \cos\left(\frac{n\pi x}{2}\right) \). Then \( (\phi_n)_{n=1}^\infty \) is an ON-basis for \( L^2([0,1]) \).

Then \( [K\phi_n](x) = \sqrt{2} \int_0^x \cos\left(\frac{n\pi x}{2}\right)dy = \left[ \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \right]_0^x = \sqrt{2} \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \).

Set \( \psi_n(x) = \sqrt{2} \sin\left(\frac{n\pi x}{2}\right) \). Then \( (\psi_n)_{n=1}^\infty \) is also an ON-basis for \( L^2([0,1]) \).

We can write \( x = \sum_{n=1}^\infty \alpha_n \phi_n \).

Then \( \|Kx\|^2 = \left\| \sum_{n=1}^\infty \alpha_n K\phi_n \right\|^2 = \left\| \sum_{n=1}^\infty \alpha_n \frac{2}{n\pi} \psi_n \right\|^2 = \sum_{n=1}^\infty |\alpha_n|^2 \left( \frac{2}{n\pi} \right)^2 \leq \frac{4}{\pi^2} \sum_{n=1}^\infty |\alpha_n|^2 = \frac{4}{\pi^2} \|x\|^2 \) so \( \|K\| \leq \frac{2}{\pi} \).

Since \( \|K\phi_1\|^2 = \frac{2}{\pi} \|\phi_1\|^2 \) we also have that \( \|K\| \geq \frac{2}{\pi} \). Together we get \( \|K\| = \frac{2}{\pi} \).

Remark: We have determined the singular value decomposition of \( K \):

\( Kx = \sum_{n=1}^\infty \frac{\sigma_n}{n\pi} \phi_n \) where \( \sigma_n = \frac{2}{n\pi} \) are the singular values.

We can then conclude that \( \|K\| = \max_n \sigma_n = \sigma_1 = \frac{2}{\pi} \).
c)  Show that the spectral radius of $K$ is equal to zero.

\[
[K^2u](x) = \int_0^x \int_0^x u(z)dzdy = \int_0^x u(z)xdz = \int_{-x}^x (x-z)u(z)dz
\]

\[
[K^3u](x) = \int_0^x \int_0^x (y-z)u(z)dzdy = \int_0^x u(z)x^2dydz = \int_0^x \frac{(x-z)^2}{2}u(z)dz
\]

This generalizes to $[K^n u](x) = \cdots = \int_0^x \frac{(x-z)^{n-1}}{(n-1)!}u(z)dz$

So $\|K^n u\|^2 = \int_0^x \left( \int_0^x \frac{(x-z)^{n-1}}{(n-1)!}u(z)dz \right)^2 dx \leq \frac{1}{c^2(n-1)!} \int_0^x \left( \int_0^x (x-z)^{2(n-1)}dz \right)^2 \left( \int_0^x u^2(y)dy \right)dx \leq \frac{\|u\|^2}{(n-1)!}$

This implies that $\|K^n\|^2 \leq \frac{1}{(n-1)!}$, so $r(K) = \lim_{n \to \infty} \left( \frac{1}{(n-1)!} \right)^{1/n} = 0$

\[d) \quad \text{Show that } 0 \text{ belongs to the continuous spectrum of } K.\]

Set $Ku = v$.

Pick $\tilde{v} \in P$ s.t. $\|v - \tilde{v}\| < \varepsilon$ where $P$ is the set of functions $(\sin(nx))_{n=1}^\infty$. We have previously shown that this is a basis.

Set $\tilde{u} = \tilde{v}'$ then $\tilde{u} \in L^2(I)$ and $[K\tilde{u}](x) = \int_0^x \tilde{v}(y)dy = \tilde{v}(x) - \tilde{v}(0) = \tilde{v}(x)$

The final equality uses the fact that $\sin(0) = 0$.

So $\tilde{v}(x) \in \text{ran} K$ and $\|v - \tilde{v}\| < \varepsilon$, hence $0$ is in the continuous spectrum.
9.8) Define the right shift operator $S$ on $l^2(Z)$ by $S(x)_k = x_{k-1}$ $\forall k \in Z$ where $x = (x_k)_{k=-\infty}^\infty$ is in $l^2(Z)$. Prove the following (a-d).

First recall the Fourier transform: $F^{-1}x = \sum_{n=-\infty}^{\infty} x_n \frac{e^{int}}{\sqrt{2\pi}}$

Set $\mathcal{S} = F^{-1}SF$, then $\mathcal{S} - \lambda I = F^{-1}SF - \lambda F^{-1}F = F^{-1}(S - \lambda I)F$

Now $\lambda \in \sigma_\alpha(S) \iff \lambda \in \sigma_\alpha(\mathcal{S})$, $\alpha = p, c, r$

Then $F^{-1}Sx = \sum_{n=-\infty}^{\infty} x_{n-1} \frac{e^{int}}{\sqrt{2\pi}} = e^\mu \sum_{n=-\infty}^{\infty} x_{n-1} \frac{e^{in(t-\mu)}}{\sqrt{2\pi}} = e^\mu \bar{x}(t)$ so $[\mathcal{S}\bar{x}](t) = e^\mu \bar{x}(t)$

Assume $|\lambda| \neq 1$. Given $\mathcal{y} \in L^2(T)$ we have $(S - \lambda I)\frac{1}{e^\mu - \lambda} \mathcal{y}(t) = \mathcal{y}(t)$ so $(S - \lambda I)$ is bijective.

$(S - \lambda I)$ bijective implies $\lambda \in \rho(\mathcal{S})$ (*)

Assume $|\lambda|=1$ and $(\mathcal{S} - \lambda I)\bar{x} = 0$. Then $(e^\mu - \lambda I)\bar{x}(t) = 0$ almost everywhere which implies $\bar{x} = 0$.

This means that $(\mathcal{S} - \lambda I)$ is one-to-one, so we can immediately conclude that $\lambda \notin \sigma_\rho(\mathcal{S})$. (**)

Note that $1 \notin \text{ran}(\mathcal{S} - \lambda I) \Rightarrow \text{ran}(\mathcal{S} - \lambda I) \neq L^2(T)$ (***)

However, given a $\mathcal{y} \in L^2(T)$ set $\mathcal{y}_m(t) = \begin{cases} \mathcal{y}(t) & |\lambda - e^\mu| \geq \frac{1}{n} \\ 0 & \text{else} \end{cases}$ then $\mathcal{y}_m(t) \rightarrow y(t)$ and $\mathcal{y}_m \in \text{ran}(\mathcal{S} - \lambda I)$ since $(\mathcal{S} - \lambda I)\mathcal{y}_m(t) = \mathcal{y}_m(t)$ (****)

a) The point spectrum of $S$ is empty.

The equations (*) and (***) above show that $\lambda$ isn’t in the point spectrum for $|\lambda| \neq 1$ and $|\lambda|=1$ respectively. Combined they show that the point spectrum is empty.

b) $\text{ran}(\lambda I - S) = l^2(Z)$ for every $\lambda \in C$ with $|\lambda| > 1$

Equation (*) above shows this.

c) $\text{ran}(\lambda I - S) = l^2(Z)$ for every $\lambda \in C$ with $|\lambda| < 1$

Equation (*) above shows this.

d) The spectrum of $S$ consists of the unit circle $\{\lambda \in C \mid |\lambda| = 1\}$ and is purely continuous.

Equation (*) shows that $\lambda$ is not in the spectrum for $|\lambda| \neq 1$. Equations (***) and (****) combine to show that all $\lambda$ with $|\lambda| = 1$ are in the continuous spectrum.
Define the discrete Laplacian operator $\Delta$ on $l^2(Z)$ by $(\Delta x)_k = x_{k-1} - 2x_k + x_{k+1}$, where $x = (x_k)_{k=-\infty}^{\infty}$. Show that $\Delta = S + S^* - 2I$ and prove that the spectrum of $\Delta$ is entirely continuous and consists of the interval $[-4,0]$.

Noting that on $l^2(Z)$ the adjoint of the right shift operator is the left shift operator (see problem 3 of homework 3), the fact that $\Delta = S + S^* - 2I$ follows directly.

**Spectrum:** As we did in the previous problem we begin by switching to the Fourier domain. Then

$$F^{-1}\Delta x = \sum_{n=-\infty}^{\infty} (x_{n-1} + x_{n+1} + 2x_n) \frac{e^{i\pi n}}{\sqrt{2\pi}} = e^{i\pi} \sum_{n=-\infty}^{\infty} x_{n-1} \frac{e^{i\pi(n-1)}}{\sqrt{2\pi}} + e^{-i\pi} \sum_{n=-\infty}^{\infty} x_{n+1} \frac{e^{i\pi(n+1)}}{\sqrt{2\pi}} + 2\sum_{n=-\infty}^{\infty} x_n \frac{e^{i\pi n}}{\sqrt{2\pi}} = (e^{i\pi} + e^{-i\pi} + 2) \hat{x}(t)$$

so $\Delta \hat{x}(t) = (e^{i\pi} + e^{-i\pi} + 2) \hat{x}(t)$

Note that $e^{i\pi} + e^{-i\pi} + 2 \leq \sup |e^{i\pi}| + \sup |e^{-i\pi}| + 2 \leq 4$

Assume $|\lambda| > 4$. Given $\tilde{y} \in L^2(T)$ we have \((\Delta - \lambda I) \frac{1}{(e^{i\pi} + e^{-i\pi} + 2) - \lambda} \tilde{y}(t) = \tilde{y}(t)\) so $(\Delta - \lambda I)$ is bijective.

Note that $e^{i\pi} + e^{-i\pi} + 2 \geq -\sup |e^{i\pi}| - \sup |e^{-i\pi}| + 2 \geq 0$

Assume $|\lambda| < 4$. Given $\tilde{y} \in L^2(T)$ we have \((\Delta - \lambda I) \frac{1}{(e^{i\pi} + e^{-i\pi} + 2) - \lambda} \tilde{y}(t) = \tilde{y}(t)\) so $(\Delta - \lambda I)$ is bijective.

So the spectrum consists of the interval $[-4,0]$. We just need to show that it is continuous.

**Continuous:** Note that $1 \notin \text{ran}(\Delta - \lambda I) \Rightarrow \text{ran}(\Delta - \lambda I) \notin L^2(T)$

However, given a $\tilde{y} \in L^2(T)$ set $\tilde{y}_m(t) = \begin{cases} \tilde{y}(t) & \frac{|\lambda| - (e^{i\pi} + e^{-i\pi} + 2)}{1/n} \geq 1/n \\ \text{else} \end{cases}$ then $\tilde{y}_m(t) \rightarrow \tilde{y}(t)$ as $m \rightarrow \infty$

and $\tilde{y}_m \in \text{ran}(\Delta - \lambda I)$ since $(\Delta - \lambda I) \frac{\tilde{y}_m(t)}{(e^{i\pi} + e^{-i\pi} + 2) - \lambda} = \tilde{y}_m(t)$
9.11) The approximate spectrum is defined \( \sigma_{\text{app}}(A) = \{ \lambda : \exists (x_n) \text{ s.t. } \| x_n \| = 1 \text{ and } \| (A - \lambda I)x_n \| \to 0 \} \).

Prove the following:

(a) \( \sigma_{\text{app}}(A) \subseteq \sigma(A) \)

(b) \( \sigma_p(A) \subseteq \sigma_{\text{app}}(A) \)

(c) \( \sigma_c(A) \subseteq \sigma_{\text{app}}(A) \)

(d) Give an example to show that a point in the residual spectrum need not belong to the approximate spectrum.

a) Prove \( \sigma_{\text{app}}(A) \subseteq \sigma(A) \)

Assume \( \lambda \in \sigma(A)^c = \rho(A) \). Then \( (A - \lambda I)^{-1} \) is a bounded operator. If \( (x_n) \) is any sequence of vectors with \( \| x_n \| = 1 \) then set \( y_n = (A - \lambda I)x_n \).

Then \( 1 = \| x_n \| = \| (A - \lambda I)^{-1} y_n \| \leq \| (A - \lambda I)^{-1} \| \cdot \| y_n \| \).

Also \( \| y_n \| = \| (A - \lambda I)x_n \| \geq \frac{1}{\| (A - \lambda I)^{-1} \|} \) so \( \lambda \notin \sigma_{\text{app}}(A) \).

b) Prove \( \sigma_p(A) \subseteq \sigma_{\text{app}}(A) \)

Assume \( \lambda \in \sigma_p(A) \). Then there exists an \( x \neq 0 \) s.t. \( Ax = \lambda x \). Set \( x_n = \frac{x}{\| x \|} \), then \( \| (A - \lambda I)x_n \| = 0 \) so \( \lambda \in \sigma_{\text{app}}(A) \).

c) Prove \( \sigma_c(A) \subseteq \sigma_{\text{app}}(A) \)

Assume \( \lambda \in \sigma_c(A) \). Then \( \text{ran}(A - \lambda I) = H \). Set \( \alpha = \inf_{\| x \| = 1} \| (A - \lambda I)x \| \). We want to prove that \( \alpha = 0 \) (if it is then we can pick \( x_n \) s.t. \( \| x_n \| = 1 \) and \( \| (A - \lambda I)x_n \| \to 0 \)).

If \( \alpha \neq 0 \) then by Proposition 5.30 \( \text{ran}(A - \lambda I) \) is closed. This is impossible since \( (A - \lambda I) \) is not onto but \( \text{ran}(A - \lambda I) = H \).

d) Give an example of an operator \( A \) and a point \( \lambda \in \sigma_c(A) \) s.t. \( \lambda \notin \sigma_{\text{app}}(A) \).

Consider the right-shift operator \( S \) from question 9.10 and the point \( \lambda = 0 \). Then if \( \| x_n \| = 1 \) we have \( \| (S - \lambda I)x_n \| = \| Sx_n \| = \| x_n \| = 1 \).