APPM5450 — Applied Analysis: Section exam 1
8:30 – 9:50, February 10, 2014. Closed books.

The following problems are worth 25 points each.

Problem 1: Consider the Hilbert space $H = \ell^2(\mathbb{N})$, and the operator

$$A(x_1,\, x_2,\, x_3,\, \dots) = (\lambda_1 x_1,\, \lambda_2 x_2,\, \lambda_3 x_3,\, \dots),$$

where $(\lambda_n)_{n=1}^{\infty}$ is a bounded sequence of complex numbers.

(a) Prove that $||A|| = \sup_n |\lambda_n|$.

(b) Give minimal conditions on the numbers λ_n that ensure that A is, respectively:
 (i) self-adjoint,
 (ii) non-negative,
 (iii) positive,
 (iv) coercive.
 Motivate your claims.

Problem 2: Let $\mathbb{T}$ denote the unit circle parameterized using the interval $I = [-\pi, \pi]$ as usual, and define the function $f \in L^2(\mathbb{T})$ via

$$f(x) = \begin{cases} 1 & \text{when } |x| \le \pi/2, \\ 0 & \text{when } |x| > \pi/2. \end{cases}$$

(a) Compute the Fourier series of f.
(b) Determine for which $s \in \mathbb{R}$ it is the case that f belongs to the Sobolev space $H^s(\mathbb{T})$.
(c) Now define a function $g \in L^2(\mathbb{T}^2)$ via

$$g(x_1, x_2) = f(x_1) f(x_2).$$

 For which $s \in \mathbb{R}$ can you say for sure that $g \notin H^s(\mathbb{T}^2)$?

Problem 3: Set $I = [-1, 1]$ and let Ω denote the set of continuous functions on I, viewed as a subset of $H = L^2(I)$. Define an operator $A : \Omega \to L^2(I)$ via

$$[Au](x) = \frac{1}{2} u(x) + \frac{1}{2} u(-x).$$

(a) Prove that A can be uniquely extended to an operator in $\mathcal{B}(H)$.

(b) Is A a projection? If yes, is it an orthogonal projection?

Motivate your answers.

Problem 4: Let $(e_n)_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H, and let $\mathcal{P}$ denote the set of all ***finite*** linear combinations of elements of e_n's. (Recall that we write this $\mathcal{P} = \text{Span}\big(e_n\big)_{n=1}^{\infty}$.) Prove that:

$$\mathcal{P} \text{ is dense} \quad \Leftrightarrow \quad \big(e_n\big)_{n=1}^{\infty} \text{ is an ON-basis.}$$