APPM5450 — Applied Analysis: Section exam 3
8:30 – 9:50, April 23, 2014. Closed books.

Problem 1: (20p) Let m denote Lebesgue measure on $\mathbb{R}$, and let $\mathcal{L}$ denote the set of Lebesugue measurable sets, as usual. Which of the following functions n are ***not*** measures on $(\mathbb{R}, \mathcal{L})$:

(a) $n(\Omega) = 2m(\Omega)$
(b) $n(\Omega) = (m(\Omega))^2$
(c) $n(\Omega) = 1 + m(\Omega)$
(d) $n(\Omega) = m(2\Omega)$, where $2\Omega = \{2x : x \in \Omega\}$
(e) $n(\Omega) = \int_\Omega e^x dm(x)$
(f) $n(\Omega) = \int_\Omega \sin(x) dm(x)$

Motivate briefly, but only for those that are ***not*** measures.

Problem 2: (10p) For which of the following functions is the Lebesgue integral defined w.r.t. Lebesgue measure on the specified set X. (No motivation necessary.)

(a) $f(x) = \sin(x)$ on $X = \mathbb{R}$.

(b) $f(x) = e^x$ on $X = \mathbb{R}$.

(c) $f(x) = 1/x$ (with $f(0) = 0$) on $X = (-1, 1)$.

(d) $f(x) = 1/x$ on $X = (0, \infty)$.

Problem 3: (30p) The following are worth 10 points each.

(a) Evaluate $\displaystyle\int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx$.

(b) Set $f_n(x) = \cos(nx)\,\dfrac{\sin(x/2)}{x}$. What is its Fourier transform $\hat{f}_n$?

(c) Set $h_N = \displaystyle\sum_{n=1}^{N} f_n$, where f_n is defined as in (b). Does the sequence $(h_N)_{N=1}^\infty$ converge in $\mathcal{S}^*(\mathbb{R})$? If yes, then specify the limit and motive briefly. If no, then motivate briefly.

Hint: *The function* $g = \chi_{(-1/2,1/2)}$ *has the Fourier transform* $\hat{g}(t) = \sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(t/2)}{t}$.

Problem 4: (20p) Suppose that $f \in H^3(\mathbb{R}^d)$, where d is a positive integer, and that u satisfies

$$-\Delta u + u = f \qquad \text{on } \mathbb{R}^d,$$

where Δ denotes the Laplace operator as usual,

$$\Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

For which $t \geq 0$ do we know that $u \in H^t(\mathbb{R}^d)$? Can you for any d say for sure that $u \in C_0(\mathbb{R}^d)$?

Problem 5: (20p) Below, four different distributions $T \in \mathcal{S}^*(\mathbb{R})$ are specified. In each case, give its Fourier distribution $\hat{T}$. Simplify $\hat{T}$ as far as you can, and provide a *brief* motivation.

(a) Set $T(x) = x$.

(b) Set $T(x) = x\,e^{-x^2/2}$.

(c) Fix two functions $\psi, \chi \in \mathcal{S}(\mathbb{R})$, and let T denote their convolution, $T(x) = [\psi * \chi](x)$.

(d) Set $f(x) = x$, let $S = \delta'$ denote the derivative of the Dirac δ-function, and define the product of f and S as $T = f\,S$.