APPM5450 — Applied Analysis: Section exam 3 — Solutions
8:30 – 9:50, April 23, 2014. Closed books.

Problem 1: (20p) Let m denote Lebesgue measure on $\mathbb{R}$, and let $\mathcal{L}$ denote the set of Lebesugue measurable sets, as usual. Which of the following functions n are ***not*** measures on $(\mathbb{R}, \mathcal{L})$:

(a) $n(\Omega) = 2m(\Omega)$
(b) $n(\Omega) = (m(\Omega))^2$
(c) $n(\Omega) = 1 + m(\Omega)$
(d) $n(\Omega) = m(2\Omega)$, where $2\Omega = \{2x : x \in \Omega\}$
(e) $n(\Omega) = \int_\Omega e^x dm(x)$
(f) $n(\Omega) = \int_\Omega \sin(x) dm(x)$

Motivate briefly, but only for those that are ***not*** measures.

Solution:

(b) is not a measure since it is not additive. Set, e.g, $\Omega_1 = (0, 1)$ and $\Omega_2 = (1, 2)$. Then $n(\Omega_1) + n(\Omega_2) = 1^2 + 1^2 = 2$, but $n(\Omega_1 \cup \Omega_2) = (1 + 1)^2 = 4$.

(c) is not a measure since $n(\emptyset) = 1$.

(f) is not a measure since it can take on negative values. For instance $n((\pi, 2\pi)) = -2$.

Problem 2: (10p) For which of the following functions is the Lebesgue integral defined w.r.t. Lebesgue measure on the specified set X. (No motivation necessary.)

(a) $f(x) = \sin(x)$ on $X = \mathbb{R}$.

(b) $f(x) = e^x$ on $X = \mathbb{R}$.

(c) $f(x) = 1/x$ (with $f(0) = 0$) on $X = (-1, 1)$.

(d) $f(x) = 1/x$ on $X = (0, \infty)$.

Solution:

(a) and (c) do not have a well-defined integrals since $\int f_+ = \int f_- = \infty$.

Problem 3: (30p) The following are worth 10 points each.

(a) Evaluate $\int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx.$

(b) Set $f_n(x) = \cos(nx)\,\frac{\sin(x/2)}{x}$. What is its Fourier transform $\hat{f}_n$?

(c) Set $h_N = \sum_{n=1}^{N} f_n$, where f_n is defined as in (b). Does the sequence $(h_N)_{N=1}^\infty$ converge in $\mathcal{S}^*(\mathbb{R})$? If yes, then specify the limit and motive briefly. If no, then motivate briefly.

Hint: *The function* $g = \chi_{(-1/2,1/2)}$ *has the Fourier transform* $\hat{g}(t) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(t/2)}{t}$.

Solution:

(a) The idea here is to observe that the integral is essentially the L^2 norm of the function $\hat{g}$ specified in the hint. Then observe that by the Parceval equality, $||\hat{g}||_{L^2} = ||g||_{L^2}$, and $||g||_{L^2}$ is trivially seen to equal 1. To be precise,

$$\int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx = \int_0^\infty \left(\frac{\sin(y/2)}{y/2}\right)^2 dy/2 = 2\int_0^\infty \left(\frac{\sin(y/2)}{y}\right)^2 dy$$
$$= \int_{-\infty}^\infty \left(\frac{\sin(y/2)}{y}\right)^2 dy = ||\sqrt{\frac{\pi}{2}}\,\hat{g}||_{L^2}^2 = \frac{\pi}{2}\,||\hat{g}||_{L^2}^2 = \frac{\pi}{2}.$$

(b) First write out $\cos(nx)$ as exponentials:

$$f_n(x) = \cos(nx)\sqrt{\pi/2}\hat{g}(x) = (1/2)(e^{inx} + e^{-inx})\sqrt{\pi/2}\hat{g}(x).$$

Now use that multiplication by an exponential corresponds to a shift in Fourier space:

$$\hat{f}_n(x) = (1/2)(\tau_{-n} + \tau_n)\sqrt{\pi/2}\hat{\hat{g}}(x) = (1/2)(\tau_{-n} + \tau_n)\sqrt{\pi/2}g(x),$$

since $\mathcal{F}^2 = R$, and $Rg = g$. To summarize,

$$\hat{f}_n = \sqrt{\pi/8}\left(\chi_{(-n-1/2,-n+1/2)} + \chi_{(n-1/2,n+1/2)}\right).$$

(c) Use that (h_N) converges iff $(\hat{h}_N)$ converges (since $\mathcal{F}$ is an isomorphism on $\mathcal{S}^*$). We easily find that

$$\hat{h}_N \to \sqrt{\pi/8}\left(\chi_{(-\infty,-1/2)} + \chi_{(1/2,\infty)}\right) = \sqrt{\pi/8}\left(1 - \chi_{(-1/2,1/2)}\right).$$

Applying $\mathcal{F}^*$, we find that $h_N \to h$, where

$$h(x) = \sqrt{\frac{\pi}{8}}\left(\sqrt{2\pi}\delta - \hat{g}(x)\right) = \sqrt{\frac{\pi}{8}}\left(\sqrt{2\pi}\delta - \sqrt{\frac{2}{\pi}}\,\frac{\sin(x/2)}{x}\right) = \frac{\pi}{2}\,\delta - \frac{\sin(x/2)}{2x}.$$

Problem 4: (20p) Suppose that $f \in H^3(\mathbb{R}^d)$, where d is a positive integer, and that u satisfies

$$-\Delta u + u = f \qquad \text{on } \mathbb{R}^d,$$

where Δ denotes the Laplace operator as usual,

$$\Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

For which $t \geq 0$ do we know that $u \in H^t(\mathbb{R}^d)$? Can you for any d say for sure that $u \in C_0(\mathbb{R}^d)$?

Solution: We see that $\hat{u}(t) = (1 + |t|^2)^{-1}\,\hat{f}(t)$, which immediately implies that $u \in H^{3+2}(\mathbb{R}^d) = H^5(\mathbb{R}^d)$. Since the Sobolev spaces are nested, we find: $\boxed{u \in H^t \text{ for } 0 \leq t \leq 5}$

Recall that $u \in H^s(\mathbb{R}^d)$ implies that $u \in C_0(\mathbb{R}^d)$ provided that $s > d/2$. Here $s = 5$, so $\boxed{u \in C_0 \text{ provided that } d < 10.}$

Problem 5: (20p) Below, four different distributions $T \in \mathcal{S}^*(\mathbb{R})$ are specified. In each case, give its Fourier distribution $\hat{T}$. Simplify $\hat{T}$ as far as you can, and provide a *brief* motivation.

(a) Set $T(x) = x$.

(b) Set $T(x) = x\,e^{-x^2/2}$.

(c) Fix two functions $\psi, \chi \in \mathcal{S}(\mathbb{R})$, and let T denote their convolution, $T(x) = [\psi * \chi](x)$.

(d) Set $f(x) = x$, let $S = \delta'$ denote the derivative of the Dirac δ-function, and define the product of f and S as $T = f\,S$.

Solution:

(a) We have $\mathcal{F}^*(\delta) = \beta$ where $\beta = 1/\sqrt{2\pi}$. Then $\mathcal{F}^*(\delta')(x) = -ix\beta = -i\beta T(x)$. Then $\delta' = -i\beta\hat{T}$ and so $\hat{T} = i\sqrt{2\pi}\delta'$.

(b) Set $g(x) = e^{-x^2/2}$. Recall that $\hat{g} = g$. Then

$$\mathcal{F}[xe^{-x^2/2}](t) = -\mathcal{F}[g'](t) = -it\,\hat{g}(t) = -it\,g(t) = -it\,e^{-t^2/2}.$$

(c) According to a theorem in the syllabus, $\hat{T}(t) = \sqrt{2\pi}\,\hat{\psi}(t)\,\hat{\chi}(t)$.

(d) First we simplify T. Note that $\langle T, \varphi\rangle = \langle \delta', f\varphi\rangle = -f'(0)\varphi(0) - f(0)\varphi'(0) = -\varphi(0)$, so $T = -\delta$. Then $\hat{T} = -1/\sqrt{2\pi}$.