UNITARY MAPS – A GENERALIZATION OF ORTHOGONAL MATRICES

Def^n Let H_1 & H_2 be two Hilbert Spaces.

An operator $U : H_1 \to H_2$ is said to be UNITARY if it is bijective, and if

$$(Ux, Uy)_{H_2} = (x, y)_{H_1} \qquad \forall x, y \in H_1$$

In other words, a unitary map is a Hilbert Space Isomorphism.

What happens if $H_1 = H_2$?

$$U \text{ is unitary} \iff (Ux, Uy) = (x, y) \quad \forall x, y$$
$$\iff (U^*Ux, y) = (x, y) \quad \forall x, y$$

So if U is unitary, then $U^*U = I$.

However, it is not the case that $U^*U = I \Rightarrow U$ unitary since U must also be bijective.

<u>Counterexample</u> Let R denote the right-shift operator.

R is one-to-one, but not onto.

However, $R^*R = I$.

So we must require <u>both</u> that U is invertible, and that $U^*U = I$.

<u>Lemma</u> Let H be a H.S. and let $U \in B(H)$.

Then U is unitary $\iff$ U is invertible, and $U^{-1} = U^*$.

Note that if $U \in B(H)$ is unitary, then both $UU^* = I$ and $U^*U = I$ (this is not true for the right-shift operator!)

<u>Example</u> $H = \ell^2(\mathbb{Z})$ ← doubly infinite sequences.

so $x \in H \Leftrightarrow x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ and $\sum_{j=-\infty}^{\infty} |x_j|^2 < \infty$

Let R denote the rightshift operator on H.

$[Rx]_j = x_{j-1}$

Then $R^* = L$, the left-shift operator, and $R^{-1} = R^* = L$

so R <u>is</u> a unitary operator.

<u>Example</u> Let A be a bdd S-A operator on a H.S. H.

Set $B = \exp(iA) = \sum_{n=0}^{\infty} \frac{1}{n!} (iA)^n$

Then $B^* = \left(\sum_{n=0}^{\infty} \frac{1}{n!} (iA)^n \right)^* = \sum_{n=\cancel{\infty}\,0}^{\infty} \frac{1}{n!} (-iA^*)^n$

$\underset{\text{Use } A^* = A}{=} \sum_{n=\cancel{\infty}\,0}^{\infty} \frac{1}{n!} (-iA)^n = \exp(-iA) = B^{-1}$

So B is a unitary map.

(Recall that if $\lambda \in \mathbb{R}$ then $|e^{i\lambda}| = 1$ which is the analogous result in the 1-dim Hilbert space $\mathbb{C}$.)

Example Let H_1 be a H.S. with an ON-basis $(\phi_n)_{n=-\infty}^{\infty}$.
Let H_2 be a H.S. with an ON-basis $(\psi_n)_{n=-\infty}^{\infty}$
Then the map $U: H_1 \to H_2 : x \mapsto \sum_{n=-\infty}^{\infty} (\phi_n, x)\psi_n$
is a unitary map.
This is the "change of coordinate" map.
For instance, the Fourier transform is of this type with

$H^1 = L^2(\mathbb{T})$ $\qquad \phi_n = \dfrac{e^{int}}{\sqrt{2\pi}}$

$H^2 = \ell^2(\mathbb{Z})$ $\qquad \psi_n = (\dots 0, 0, \overset{n\text{'th position}}{1}, 0, 0, \dots)$

$f \in L^2(\mathbb{T}) \Rightarrow Uf = (\alpha_n)_{n=-\infty}^{\infty}$

where $\alpha_n = (\phi_n, f) = \displaystyle\int_{\mathbb{T}} \frac{e^{-int}}{\sqrt{2\pi}} f(t)\,dt$