

Homework set 10 — APPM5450, Spring 2017

From the textbook: 11.18, 11.13, 11.16.

In 11.16, you're free to assume that f is smooth (or that $f \in \mathcal{S}(\mathbb{R}^3)$), if you like.

Problem 1: Let R denote a real number such that $0 < R < \infty$ and define

$$f_n(x) = \begin{cases} n \cos(nx) & \text{for } |x| \leq R, \\ 0, & \text{for } |x| > R. \end{cases}$$

For which numbers R , if any, is it the case that $f_n \rightarrow 0$ in \mathcal{S}^* ?

Problem 2: (Optional review of old material.) Prove that $C_c(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$. Prove that $C_0(\mathbb{R}^d)$ is a closed subset of $C_b(\mathbb{R}^d)$.

What follows is a set of review questions for Chapter 11. They are not part of the home work but you may find them useful in preparing for the third midterm and the final:

What does it mean for $\varphi_n \rightarrow \varphi$ in \mathcal{S} ?

Make absolutely sure that you understand problems like 11.4, 11.10a.

Prove that if $\varphi_n \rightarrow \varphi$ in \mathcal{S} , then $x\varphi_n(x) \rightarrow x\varphi(x)$ and $\partial\varphi_n \rightarrow \partial\varphi$ in \mathcal{S} .

Let T be a linear map from \mathcal{S} to \mathbb{R} . What does it mean for T to be continuous? Prove that if there exists a finite C and a finite N such that $|T(\varphi)| \leq C \sum_{|\alpha|, n \leq N} \|\varphi\|_{\alpha, n}$, then T is continuous.

Let $T \in \mathcal{S}^*(\mathbb{R}^d)$, and let α be a multi-index. Define $x^\alpha T$. Prove that what you define is a tempered distribution.

Prove that $n^2 \sin(nx) \rightarrow 0$ in \mathcal{S}^* .

Is the Schwartz space dense in \mathcal{S}^* ?

Prove that $\sup_x |x^\beta \partial^\alpha \varphi(x)| < \infty \forall \alpha, \beta \iff \sup_x |(1 + |x|^2)^{k/2} \partial^\alpha \varphi(x)| < \infty \forall \alpha, k$.

Assume that $\int |f|^2 < \infty$, set $\langle T, \varphi \rangle = \int f \varphi$. Prove that $T \in \mathcal{S}^*$.

Let H be a function such that $H(x) = 1$ if $x \geq 0$, zero otherwise. Prove that $H \in \mathcal{S}^*$. Calculate H' . Let H_R denote the function that is 1 when $0 \leq x \leq R$ and zero otherwise. Prove that $H_R \rightarrow H$ in \mathcal{S}^* as $R \rightarrow \infty$.

Let ψ be a Schwartz function such that $\int \psi = 0$. Set $\varphi_n(x) = n\psi(nx)$. Does φ_n converge in \mathcal{S} ? Does φ_n converge in \mathcal{S}^* ?

2

Prove that $\text{PV}(1/x)$ is a continuous functional on \mathcal{S} .

What is the distributional derivative of $\text{PV}(1/x)$?

Define \hat{T} for $T \in \mathcal{S}^*$. Prove that what you define is a continuous map on \mathcal{S} .