

**Homework set 12 — APPM5450, Spring 2017 — Solutions**

**Problem 12.2:** We use “ $\uplus$ ” to denote disjoint unions.

(a) Suppose that  $A, B \in \mathcal{A}$ . Then note that  $A \setminus B = A \cap B^c = (A^c \cup B)^c$ . It now follows directly from the axioms that  $A \setminus B \in \mathcal{A}$ .

(b) Set  $C = B \setminus A$ . Then  $B = A \uplus C$ , so

$$\mu(B) = \mu(A \uplus C) = \mu(A) + \mu(C) \geq \mu(A).$$

(c) Set  $C = A \cap B$ . Then  $A = (A \setminus B) \uplus C$  and  $B = (B \setminus A) \uplus C$  so

$$\begin{aligned} \mu(A \cup B) &= \mu((A \setminus B) \uplus C \uplus (B \setminus A)) = \mu(A \setminus B) + \mu(C) + \mu(B \setminus A) \\ &\leq \mu(A \setminus B) + \mu(C) + \mu(C) + \mu(B \setminus A) \\ &= \mu((A \setminus B) \uplus C) + \mu(C \uplus (B \setminus A)) = \mu(A) + \mu(B). \end{aligned}$$

**Problem 12.3:** The trick is to write  $\bigcup_{n=1}^{\infty} A_n$  as a disjoint union. For  $n = 1, 2, 3, \dots$  set  $B_n = A_{n+1} \setminus A_n$ . Then

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup \left( \bigcup_{n=1}^{\infty} B_n \right),$$

where the union on the right is a disjoint one. Now use additivity twice:

$$\begin{aligned} \mu \left( \bigcup_{n=1}^{\infty} A_n \right) &= \mu \left( A_1 \cup \left( \bigcup_{n=1}^{\infty} B_n \right) \right) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N \rightarrow \infty} \left( \mu(A_1) + \sum_{n=1}^N \mu(B_n) \right) = \lim_{N \rightarrow \infty} \mu \left( A_1 \cup \left( \bigcup_{n=1}^N B_n \right) \right) = \lim_{N \rightarrow \infty} \mu(A_N). \end{aligned}$$

For the second part, set  $C = \bigcap_{n=1}^{\infty} A_n$  and  $C_n = A_n \setminus A_{n+1}$ . Then

$$\mu(A_N) = \mu \left( C \cup \left( \bigcup_{n=N}^{\infty} C_n \right) \right) = \mu(C) + \sum_{n=N}^{\infty} \mu(C_n).$$

Since  $\infty > \mu(A_1) \geq \sum_{n=1}^{\infty} \mu(C_n)$ , we find that

$$\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} \mu(C_n) = 0,$$

which completes the proof. For the counterexample, consider  $X = \mathbb{R}^2$ , and  $A_n = \{x = (x_1, x_2) : |x_2| < 1/n\}$ . Then  $\mu(A_n) = \infty$  for all  $n$ , but  $\bigcap_{n=1}^{\infty} A_n$  is the  $x_1$ -axis, which has measure zero.

**Problem 12.5:** Straight-forward.

**Problem 12.7:**

*Reflexivity:* It is obvious that  $f(x) = f(x)$  a.e.

*Symmetry:* If  $f(x) = g(x)$  a.e., then obviously  $g(x) = f(x)$  a.e.

*Transitivity:* Suppose that  $f(x) = g(x)$  a.e. and that  $g(x) = h(x)$  a.e. Set

$$A = \{x : f(x) \neq g(x)\}$$

$$B = \{x : g(x) \neq h(x)\}$$

$$C = \{x : f(x) \neq h(x)\}.$$

We know that  $\mu(A) = \mu(B) = 0$ , and we want to prove that  $\mu(C) = 0$ . It is clearly the case that  $C \subseteq A \cup B$ , and then it follows directly that  $\mu(C) \leq \mu(A) + \mu(B) = 0$ .