Let H be a Hilbert space, and let (x_n) be a sequence in H.

Definition: $x_n \rightharpoonup x$ means that for every $y \in H$ we have $\lim_{n\to\infty} \langle y, x_n \rangle = \langle y, x \rangle$.

1. If $x_n \rightharpoonup x$, then $\sup_n \|x_n\| < \infty$.
2. If $x_n \rightharpoonup x$, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.
3. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.
4. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal sequence. Then $e_n \rightharpoonup 0$.
5. Let (x_n) be a bounded sequence in H, and let Ω be a dense subset of H.
   If $\lim_{n\to\infty} \langle y, x_n \rangle = \langle y, x \rangle$ for every $y \in \Omega$, then $x_n \rightharpoonup x$.
6. Let (x_n) be a bounded sequence in H, and let $\{e_\alpha\}_{\alpha \in A}$ be an ON-basis for H.
   If $\lim_{n\to\infty} \langle e_\alpha, x_n \rangle = \langle e_\alpha, x \rangle$ for every $\alpha \in A$, then $x_n \rightharpoonup x$.
7. The unit ball in H is compact with respect to the weak topology.
8. Any bounded sequence in H has a weakly convergent subsequence.

Note: We already covered many of these facts in the Banach space chapter!