APPM5450 — Applied Analysis: Section exam 1
8:30am – 9:50am, February 17, 2017. Closed books.

Name:____

Problem 1: (20p) Let $H = L^2(\mathbb{T})$, and let $(u_n)_{n=1}^{\infty}$ be a sequence in H. In the chart below, we provide on each row some information about this sequence. Mark the statements that are true with a "T."

Note: The rows are independent — they do not refer to the same sequence!

	Necessarily converges weakly.	Necessarily has a weakly convergent subsequence.	Necessarily converges in norm.	Necessarily has a norm convergent subsequence.
$(u_n)_{n=1}^{\infty}$ is an orthonormal sequence.				
$(u_n)_{n=1}^{\infty}$ is a bounded sequence.				
$(u_n)_{n=1}^{\infty} \subseteq K$ where K is pre-compact in the norm topology.				
$u_n(x) = \sin(nx)$.				
$u_n(x) = n\,\sin(nx)$.				

Problem 2: (20p) Let $H = L^2(\mathbb{T})$, and suppose that for $u \in H$, you know that

$$\langle e_n,\, u\rangle = -i\,\mathrm{sign}(n)\,\sqrt{\frac{\pi}{2}}\,\frac{1}{n^2}, \qquad \text{for } n \neq 0,$$

where $e_n(t) = e^{int}/\sqrt{2\pi}$ are the elements of the standard Fourier basis. You also know that $\langle e_0,\, u\rangle = 0$. No motivation is required in the following:

(a) (10p) Specify for which $m \geq 0$ it is the case that $u \in C^m(\mathbb{T})$. *Answer:*________________________

(b) (10p) Specify for which $k \geq 0$ it is the case that $u \in H^k(\mathbb{T})$. *Answer:*________________________

Hint: You may use that $\displaystyle\sum_{n=-N}^{N} \alpha_n\,\frac{e^{int}}{\sqrt{2\pi}} = \sum_{n=1}^{N} \frac{1}{n^2}\sin(nt)$.

Problem 3: (20p) Let H be a Hilbert space and let $P \in \mathcal{B}(H)$.

(a) (5p) Specify what P must satisfy to be a *projection.*

(b) (15p) Prove that if P is a projection and $\mathrm{ran}(P) \neq \ker(P)^{\perp}$, then $\|P\| > 1$.

Problem 4: (20p) Let H be a Hilbert space, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in H, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence of complex numbers. Define $A \in \mathcal{B}(H)$ via

$$Au = \sum_{n=1}^{\infty} \lambda_n\,\langle e_n,\, u\rangle\, e_n.$$

(a) (10p) Prove that $\|A\| = \sup_{n\in\mathbb{N}} |\lambda_n|$.

(b) (10p) Which of the following statements are necessarily true:

(i) If every λ_n is real, then A is self-adjoint.
(ii) If $|\lambda_n| = 1$ for every n, then A is unitary.
(iii) Any operator A of this type is normal.
(iv) If $\lambda_n \in \{0, 1\}$ for every n, then A is a projection.