APPM5450 — Applied Analysis: Section exam 1 — Solutions

8:30am – 9:50am, February 17, 2017. Closed books.

Name:____

Problem 1: (20p) Let $H = L^2(\mathbb{T})$, and let $(u_n)_{n=1}^{\infty}$ be a sequence in H. In the chart below, we provide on each row some information about this sequence. Mark the statements that are true with a "T."

Note: The rows are independent — they do not refer to the same sequence!

	Necessarily converges weakly.	Necessarily has a weakly convergent subsequence.	Necessarily converges in norm.	Necessarily has a norm convergent subsequence.
$(u_n)_{n=1}^{\infty}$ is an orthonormal sequence.	T	T		
$(u_n)_{n=1}^{\infty}$ is a bounded sequence.		T		
$(u_n)_{n=1}^{\infty} \subseteq K$ where K is pre-compact in the norm topology.		T		T
$u_n(x) = \sin(nx)$.	T	T		
$u_n(x) = n\,\sin(nx)$.				

Comments:

Two points were deducted for each incorrect answer.

(a) This is our standard example of a sequence that is weakly convergent, but not norm convergent.

(b) This follows from Banach-Alaoglu.

(c) It is a standard fact about compact sets that any sequence has a convergent subsequence. Then just use that if the subsequence is norm convergent, it is of course also weakly convergent.

(d) This is an orthogonal and bounded sequence, so it converges weakly. To prove that it does not converge in norm, use that $\|u_n - u_m\|^2 = \|u_n\|^2 + \|u_m\|^2 = \pi + \pi$ since $\langle u_n,\, u_m\rangle = 0$ when $m \neq n$.

(e) We have $\|u_n\|^2 = n^2\pi$ so the sequence is unbounded. This means that it does not converge weakly, and cannot have a weakly convergent subsequence.

Problem 2: (20p) Let $H = L^2(\mathbb{T})$, and suppose that for $u \in H$, you know that

$$\langle e_n,\, u\rangle = -i\,\mathrm{sign}(n)\,\sqrt{\frac{\pi}{2}}\,\frac{1}{n^2}, \qquad \text{for } n \neq 0,$$

where $e_n(t) = e^{int}/\sqrt{2\pi}$ are the elements of the standard Fourier basis. You also know that $\langle e_0,\, u\rangle = 0$. No motivation is required in the following:

(a) (10p) Specify for which $m \geq 0$ it is the case that $u \in C^m(\mathbb{T})$.

(b) (10p) Specify for which $k \geq 0$ it is the case that $u \in H^k(\mathbb{T})$.

Hint: You may use that $\displaystyle\sum_{n=-N}^{N} \alpha_n \frac{e^{int}}{\sqrt{2\pi}} = \sum_{n=1}^{N} \frac{1}{n^2}\sin(nt)$.

Solution:

It is best to do (b) first and then (a).

(b) Set $\alpha_n = \langle e_n,\, u\rangle$, and let us evaluate the Sobolev norm

$$\|u\|_{H^k}^2 = \sum_{n\in\mathbb{Z}} (1+n^2)^k |\alpha_n|^2 = \sum_{n\neq 0} (1+n^2)^k \frac{\pi}{2n^4} \sim \sum_{n=1}^{\infty} n^{2k-4}.$$

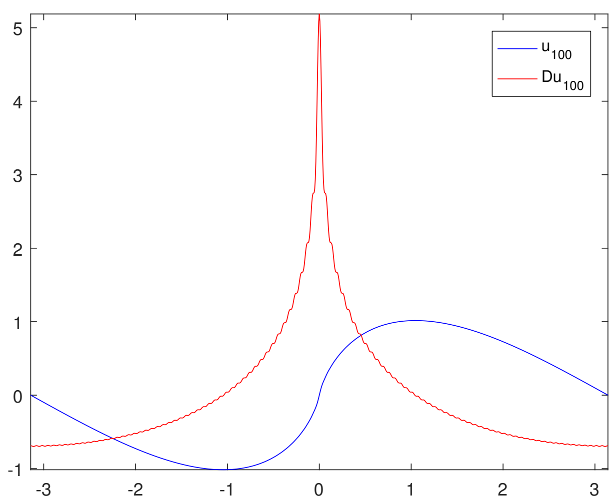
The sum is finite iff $2k - 4 < -1$, which is to say: $\boxed{\text{For } k \in [0, 3/2)}$

(a) We proved in (b) that $u \in H^k$ for some $k > 1/2$, so the Sobolev embedding theorem states that u is indeed continuous.

To check if $u \in C^1$, the Sobolev embedding theorem is not helpful. It *indicates* that u should just barely not be in C^1, but the version of the theorem that we covered does not assert this positively. However, using the hint, we can check directly. With $u_N = \sum_{n=-N}^{N} \alpha_n e_n$, we find using the hint that

$$u_N'(t) = \sum_{n=1}^{N} \frac{1}{n}\cos(nt).$$

For $t = 0$, we find that $u_N'(0) = \sum_{n=1}^{N} 1/n \sim \log(N) \to \infty$ as $N \to \infty$. This gives: $\boxed{\text{Only for } m = 0.}$



Problem 3: (20p) Let H be a Hilbert space and let $P \in \mathcal{B}(H)$.

(a) (5p) Specify what P must satisfy to be a *projection.*

(b) (15p) Prove that if P is a projection and $\mathrm{ran}(P) \neq \ker(P)^{\perp}$, then $\|P\| > 1$.

Solution:

(a) $P^2 = P$.

(b) Suppose that $\mathrm{ran}(P) \neq \ker(P)^{\perp}$. Then there are $x \in \mathrm{ran}(P)$ and $y \in \ker(P)$ such that $\langle x,\, y\rangle \neq 0$. Set $\alpha = \overline{\langle x,\, y\rangle}/|\langle x,\, y\rangle|$ and $z = \alpha y$. Then $z \in \ker(P)$ and $\langle x,\, z\rangle = |\langle x,\, y\rangle| \in \mathbb{R}_{+}$. Set

$$w = x - z\,t.$$

Then $\|Pw\| = \|x\|$, and

$$\|w\|^2 = \|x\|^2 - 2\,t\,\langle x,\, z\rangle + t^2\,\|z\|^2.$$

Set $t = \langle x,\, z\rangle/\|z\|^2$, to get $\|w\| = \|x\|^2 - \frac{(\langle x,\, z\rangle)^2}{\|z\|^2} < \|x\|^2$, which shows that $\|P\| > 1$.

Problem 4: (20p) Let H be a Hilbert space, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in H, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence of complex numbers. Define $A \in \mathcal{B}(H)$ via

$$Au = \sum_{n=1}^{\infty} \lambda_n \langle e_n, u \rangle e_n.$$

(a) (10p) Prove that $\|A\| = \sup_{n\in\mathbb{N}} |\lambda_n|$.

(b) (10p) Which of the following statements are necessarily true:
  (i) If every λ_n is real, then A is self-adjoint.
  (ii) If $|\lambda_n| = 1$ for every n, then A is unitary.
  (iii) Any operator A of this type is normal.
  (iv) If $\lambda_n \in \{0, 1\}$ for every n, then A is a projection.

Solution:

(a) Set $M = \sup_n |\lambda_n|$. First we prove that $\|A\| \leq M$. For any $u \in H$, we have

$$\|Au\|^2 = \{\text{Parseval}\} = \sum_{n=1}^{\infty} |\lambda_n \langle e_n, u \rangle|^2 \leq \sum_{n=1}^{\infty} M^2 |\langle e_n, u \rangle|^2 \leq M^2 \|u\|^2.$$

Next we prove that $\|A\| \geq M$. For any n, we have that

$$\|A\| = \sup_{\|u\|=1} \|Au\| \geq \|Ae_n\| = \|\lambda_n e_n\| = |\lambda_n|.$$

Take the supremum over n to get $\|A\| \geq \sup_n |\lambda_n| = M$.

(b) Let us discuss each question in turn:

(i) TRUE. It is easy to verify that

$$A^* u = \sum_{n=1}^{\infty} \overline{\lambda_n} \langle e_n, u \rangle e_n.$$

We see that if every λ_n is real, then $A = A^*$.

(ii) FALSE. The statement is true if $\{e_n\}$ is an ON-*basis*. If it is not, then to prove that the claim is false, pick a vector $x \neq 0$ such that $\langle e_n, x \rangle = 0$ for every n. Then $\|Ax\| = 0 < \|x\|$.

(iii) TRUE. It is easily verified that

$$AA^* x = \sum_{n=1}^{\infty} \lambda_n \overline{\lambda_n} \langle e_n, u \rangle e_n = \sum_{n=1}^{\infty} \overline{\lambda_n} \lambda_n \langle e_n, u \rangle e_n = A^* Ax.$$

(iv) TRUE. We find that

$$A^2 x = \sum_{n=1}^{\infty} \lambda_n^2 \langle e_n, u \rangle e_n.$$

If every $\lambda_n \in \{0, 1\}$, then $\lambda_n^2 = \lambda_n$ so $A^2 = A$. (The converse is also true, if any λ_n is not equal to zero or one, then $A^2 \neq A$.)