APPM5450 — Applied Analysis: Section exam 2
1:00pm – 1:50pm, March 17, 2017.

Problem 1: (10p) Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$.

(a) (5p) Define the *spectrum* $\sigma(A)$.

(b) (5p) Suppose that A is skew-adjoint and that $\|A\| = 2$. Are there any complex numbers λ for which you can say for sure that $A - \lambda I$ is one-to-one and onto?

Solution:

(b) Since A is skew-adjoint, you know that if $\mathrm{Re}(\lambda) \neq 0$, then $\lambda \notin \sigma(A)$.

Since $\|A\| = 2$, you know that if $|\lambda| > 2$, then $\lambda \in \rho(A)$.

Consequently, $A - \lambda I$ is necessarily one-to-one and onto if either $|\lambda| > 2$ or if $\mathrm{Re}(\lambda) \neq 0$.

Problem 2: (10p) Let $T \in \mathcal{S}^*(\mathbb{R})$ be defined via $T(\varphi) = \displaystyle\int_{-\infty}^{\infty} \log|x|\,\varphi(x)\,dx$. Specify the derivative of T. No motivation required.

Solution:

The derivative of T is the principal value of $1/x$.

To prove this, note that

$$\begin{aligned}
[DT](\varphi) = -T(\varphi') &= \lim_{\varepsilon\searrow 0}\left\{-\int_{-\infty}^{-\varepsilon}\log|x|\,\varphi'(x)\,dx - \int_{\varepsilon}^{\infty}\log|x|\,\varphi'(x)\,dx\right\}\\
&= \lim_{\varepsilon\searrow 0}\left\{-\left[\log|x|\varphi(x)\right]_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon}\frac{1}{x}\,\varphi(x)\,dx - \left[\log|x|\varphi(x)\right]_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty}\frac{1}{x}\,\varphi'(x)\,dx\right\}\\
&= \lim_{\varepsilon\searrow 0}\left\{-\log(\varepsilon)\varphi(-\varepsilon) + \int_{-\infty}^{-\varepsilon}\frac{1}{x}\,\varphi(x)\,dx + \log(\varepsilon)\varphi(\varepsilon) + \int_{\varepsilon}^{\infty}\frac{1}{x}\,\varphi'(x)\,dx\right\} = PV(1/x)(\varphi),
\end{aligned}$$

since $\lim_{\varepsilon\searrow 0}\log(\varepsilon)\big(\varphi(\varepsilon) - \varphi(-\varepsilon)\big) = 0$.

Problem 3: (10p) No motivations required for these two problems.

(a) (5p) Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$ be an operator that satisfies $A^2 = A = A^*$. The operator A is neither the zero or the identity operator. Specify $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, and $\sigma_{\rm r}(A)$.

(b) (5p) Let $H = L^2([0,\infty))$, and let $A \in \mathcal{B}(H)$ be defined by $[Au](x) = \arctan(x)\,u(x)$. Specify $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, and $\sigma_{\rm r}(A)$.

Solution:

(a) A is a non-trivial orthogonal projection. As shown in the homework, this means that

$$\sigma_{\rm p}(A) = \{0,\,1\}, \qquad \sigma_{\rm c}(A) = \emptyset, \qquad \sigma_{\rm r}(A) = \emptyset.$$

(a) A is a multiplication operator so $\sigma(A)$ equals the closure of the range of the function being multiplied. In this case the spectrum is purely a continuum spectrum since there are no stationary points in the range. So

$$\sigma_{\rm p}(A) = \emptyset, \qquad \sigma_{\rm c}(A) = [0, \pi/2], \qquad \sigma_{\rm r}(A) = \emptyset.$$

Problem 4: (10p) Consider the four sequences in $\mathcal{S}^*(\mathbb{R})$ given below. Specify which sequences are convergent. If the sequence is convergent, then specify the limit. No motivations required.

(a) $(T_n)_{n=1}^\infty$ where $T_n(x) = \sin(nx)$.

(b) $(T_n)_{n=1}^\infty$ where $T_n(x) = \begin{cases} n & \text{when } -1/n \leq x \leq 1/n, \\ 0 & \text{when } |x| > 1/n. \end{cases}$

(c) $(T_n)_{n=1}^\infty$ where $T_n(x) = \begin{cases} n^2 & \text{when } -1/n \leq x \leq 1/n, \\ 0 & \text{when } |x| > 1/n. \end{cases}$

(d) $(T_n)_{n=1}^\infty$ where $T_n(x) = \displaystyle\sum_{m=0}^{n} \frac{x^m}{m!}$.

Solution:

(a) $\boxed{T_n \to 0.}$ We proved this in class.

(a) $\boxed{T_n \to 2\delta.}$ We proved something very similar in class.

(c) $\boxed{\text{Divergent.}}$ You can easily prove that $\lim_{n\to\infty} T_n(\varphi) = \lim_{n\to\infty} 2n\varphi(0)$.

(d) $\boxed{\text{Divergent.}}$ We have $\lim_{n\to\infty} T_n(x) = e^x$, and e^x is not a tempered distribution. (If you'd like to prove things rigorously, consider $\varphi(x) = \exp\bigl(-(1+x^2)^{1/4}\bigr)$. Then $\varphi \in \mathcal{S}$ and $T_n(\varphi) \to \infty$.)

Problem 5: (20p) Let H denote the Hilbert space $H = \ell^2(\mathbb{Z})$. In other words, a doubly indexed vector $x = \{x(n)\}_{n=-\infty}^{\infty}$ belongs to H iff $\sum\limits_{n=-\infty}^{\infty} |x(n)|^2 < \infty$. Define $A \in \mathcal{B}(H)$ via:

$$[Ax](n) = x(n+1) - x(n-1), \qquad n \in \mathbb{Z}.$$

Let $F : L^2(\mathbb{T}) \to H$ denote the standard Fourier transform, and let F^{-1} denote its inverse. Define

$$B = F^{-1}AF$$

as an operator on $L^2(\mathbb{T})$.

(a) (5p) Determine the action of B on a function $u = u(t)$ in $L^2(\mathbb{T})$.

(b) (15p) Determine $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, and $\sigma_{\rm r}(A)$.

Solution:

(a) Consider a function $u = \sum_{-\infty}^{\infty} a_n e_n$, where $e_n(x) = e^{inx}/\sqrt{2\pi}$ as usual. Then $Fu = \{a_n\}$ and $AFu = \{a_{n+1} - a_{n-1}\}$. Then

$$[F^{-1}AFu](x) = \sum_{n=-\infty}^{\infty} \big(a_{n+1} - a_{n-1}\big)\,\frac{e^{inx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} e^{-ix}\,a_{n+1}\frac{e^{i(n+1)x}}{\sqrt{2\pi}} - \sum_{n=-\infty}^{\infty} e^{ix}\,a_{n-1}\frac{e^{i(n-1)x}}{\sqrt{2\pi}}$$
$$= \big(e^{-ix} - e^{ix}\big)u(x) = -2i\sin(x)\,u(x).$$

(b) Since A and B are unitarily equivalent, their spectra are identical. First note that

$$\langle Bu,\, v\rangle = \int_{-\pi}^{\pi} \overline{-2i\sin(x)u(x)}\,v(x)\,dx = \int_{-\pi}^{\pi} \overline{u(x)}\,2i\sin(x)\,v(x)\,dx = \langle u,\, -Bv\rangle,$$

so B is skew-adjoint. This proves that $\sigma_{\rm r}(B) = \emptyset$ and that $\sigma(B)$ is a subset of the imaginary line.

Let us first search for eigenvalues. Suppose $Bu = \lambda u$. Then

$$(-2i\sin(x) - \lambda)\,u(x) = 0, \qquad \text{a.e.}$$

Since $-2i\sin(x) - \lambda = 0$ except possibly for a set of measure zero, we find that $\sigma_{\rm p}(B) = \emptyset$.

Set $\Omega = \{ib :\ b \in [-2, 2]\}$. In other words, Ω is the range of the function $f(x) = -2i\sin(x)$, and our *guess* at this point is that Ω is the continuum spectrum.

Suppose that $\lambda \notin \Omega$. Set $d = \inf\{|\lambda - z| :\ z \in \Omega\} = \text{dist}(\lambda, \Omega)$. Since Ω is closed we know that $d > 0$. Then

$$\big\|(B - \lambda I)^{-1}u\big\|^2 = \int_{-\pi}^{\pi} \left|\frac{1}{f(x) - \lambda}u(x)\right|^2 dx \le \int_{-\pi}^{\pi} \frac{1}{d^2}\,|u(x)|^2\,dx = \frac{1}{d^2}\|u\|^2$$

so $\|(B - \lambda I)^{-1}\| \le 1/d < \infty$, which shows that $\lambda \in \rho(B)$.

Suppose that $\lambda = ib \in \Omega$ for some $b \in [-\pi, \pi]$. Let $a \in [-\pi, \pi]$ be such that $f(a) = ib$. Then pick non-negative functions φ_n such that $\|\varphi_n\| = 1$, and $\varphi_n(x) = 0$ when $|x - a| \ge 1/n$. Then

$$\|(B - \lambda I)\varphi_n\|^2 = \int_{-\pi}^{\pi} |(f(x) - ib)\varphi_n(x)|^2\,dx = \int_{a-1/n}^{a+1/n} |f(x) - ib|^2\,|\varphi_n(x)|^2\,dx \le= \frac{8}{3n^3}\|\varphi_n\|^2 = \frac{8}{3n^3},$$

where we used that $|f(x) - ib| = |\int_a^x f'(x)dx| \le 2|x - a|$ since $|f'| \le 2$. The inequality proven shows that $B - \lambda I$ is not coercive, and consequently cannot have closed range.

$$\boxed{\sigma_{\rm p}(A) = \emptyset, \quad \sigma_{\rm c}(A) = \{ib :\ b \in [-2, 2]\}, \qquad \sigma_{\rm r}(A) = \emptyset.}$$

Problem 6: (4×5p) For each of the four operators defined below, determine whether it is well-defined, and whether it is continuous.

(a) $A:\,\mathcal{S}(\mathbb{R}) \to \mathbb{C}$ defined via $A(\varphi) = \displaystyle\int_{\mathbb{R}} x^2\,\varphi(x)\,dx$.

(b) $B:\,\mathcal{S}(\mathbb{R}) \to \mathbb{C}$ defined via $B(\varphi) = \displaystyle\int_{\mathbb{R}} x\,\big(\varphi(x)\big)^2\,dx$.

(c) $C:\,\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined via $[C(\varphi)](x) = x\,\varphi(x)$.

(d) $D:\,\mathcal{S}^*(\mathbb{R}) \to \mathcal{S}^*(\mathbb{R})$ defined via $DT = \partial T$. (Just plain differentiation.)

Solution:

(a) Pick $\varphi \in \mathcal{S}$. Then $|A(\varphi)| \leq \displaystyle\int \frac{x^2}{(1+x^2)^2}\,(1+x^2)^2|\varphi(x)|\,dx \leq \int \frac{x^2}{(1+x^2)^2}\,dx\|\varphi\|_{0,4} = C\|\varphi\|_{0,4}$. This proves that A is well-defined. Next we prove continuity. Suppose that $\varphi_n \to \varphi$ in $\mathcal{S}$. Then $|A(\varphi) - A(\varphi_n)| \leq C\|\varphi - \varphi_n\|_{0,4} \to 0$.

(b) Pick $\varphi \in \mathcal{S}$. Then $|B(\varphi)| \leq \displaystyle\int \frac{|x|}{(1+x^2)^2}\,\big((1+x^2)\varphi(x)\big)^2\,dx \leq \int \frac{|x|}{(1+x^2)^2}\,dx\|\varphi\|_{0,2}^2 = C\|\varphi\|_{0,2}^2$. This proves that B is well-defined. Next we prove continuity. Suppose that $\varphi_n \to \varphi$ in $\mathcal{S}$. Set $M = \sup_n \|\varphi_n\|_{0,0}$. Since (φ_n) is convergent to φ wrt the uniform norm, we know that $M < \infty$ and that $\|\varphi\|_{0,0} \leq M$. Then

$$\begin{aligned}
|B(\varphi) - B(\varphi_n)| &\leq \int_{-\infty}^{\infty} |x|\,|(\varphi(x))^2 - (\varphi_n(x))^2|\,dx = \int_{-\infty}^{\infty} |x|\,|(\varphi(x) + \varphi_n(x))(\varphi(x) - \varphi_n(x))|\,dx \\
&\leq \int_{-\infty}^{\infty} |x|\,2M\,|\varphi(x) - \varphi_n(x)|\,dx = 2M\int_{-\infty}^{\infty} \frac{|x|}{(1+x^2)^2}\,(1+x^2)^2|\varphi(x) - \varphi_n(x)|\,dx \\
&\leq 2M\int_{-\infty}^{\infty} \frac{|x|}{(1+x^2)^2}\,dx\,\|\varphi - \varphi_n\|_{0,4} \to 0.
\end{aligned}$$

(c) Fix $\varphi \in \mathcal{S}$. Fix $\alpha, k \in \mathbb{Z}_+$. Then

$$\|C(\varphi)\|_{\alpha,k} = \sup_x (1+x^2)^{k/2}|\partial^\alpha(x\varphi)| = \sup_x (1+x^2)^{k/2}|x\partial^\alpha\varphi + \alpha\partial^{\alpha-1}\varphi| \leq M\|\varphi\|_{\alpha,k+1} + \alpha\|\varphi\|_{\alpha-1,k},$$

where M is the finite number given by $M = \sup \frac{|x|(1+x^2)^{k/2}}{(1+x^2)^{(k+1)/2}}$. This inequality proves that $C(\varphi) \in \mathcal{S}$. Next consider continuity. Suppose that $\varphi_n \to \varphi$ in $\mathcal{S}$. Then for any $\alpha, k \in \mathbb{Z}_+$ we have

$$\|C(\varphi) - C(\varphi_n)\|_{\alpha,k} \leq \cdots \leq M\|\varphi - \varphi_n\|_{\alpha,k+1} + \alpha\|\varphi - \varphi_n\|_{\alpha-1,k} \to 0.$$

(d) Fix $T \in \mathcal{S}^*$. We will first prove that $D(T)$ is a distribution. Fix $\varphi \in \mathcal{S}$. Then by definition

$$\langle D(T),\,\varphi\rangle = -\langle T,\,\varphi'\rangle.$$

We proved in class that $\varphi' \in \mathcal{S}$ so $D(T)$ evaluates to a finite complex number. To establish that $D(T)$ is in $\mathcal{S}^*$, we also need to prove that $D(T)$ is continuous. This follows from the fact that $\varphi_n \to \varphi$ in $\mathcal{S}$ implies that $\varphi_n' \to \varphi'$ in $\mathcal{S}$ (also proven in class). So $D(T)$ is well-defined.

Is the map $D:\,\mathcal{S}^*(\mathbb{R}) \to \mathcal{S}^*(\mathbb{R})$ continuous? We need to prove that if $T_n \to T$ in $\mathcal{S}^*$, then $D(T_n) \to D(T)$ in $\mathcal{S}^*$. Suppose that $T_n \to T$ in $\mathcal{S}^*$. Fix $\varphi \in \mathcal{S}$. Then

$$\langle D(T_n),\,\varphi\rangle = -\langle T_n,\,\varphi'\rangle \to \{\text{Since } T_n \to T \text{ and } \varphi' \in \mathcal{S}\} \to -\langle T,\,\varphi'\rangle = \langle D(T),\,\varphi\rangle.$$

In summary: All maps are well-defined and continuous.