APPM5450 — Applied Analysis: Section exam 3 — Solutions

1:00pm – 1:40pm, April 28, 2017. Closed books.

Problem 1: (6p) Consider the function $f_n(x) = \big(1/x\big)^{1/3}\chi_{[1,n]}$ so that

$$f_n(x) = \begin{cases} x^{-1/3} & \text{when } x \in [1,n], \\ 0 & \text{when } x \notin [1,n]. \end{cases}$$

For which $p \in [1,\infty]$ does $(f_n)_{n=1}^{\infty}$ form a Cauchy sequence in $L^p(\mathbb{R})$?

Solution:

The case $p = \infty$: Suppose $N \le m < n$. Then

$$\|f_m - f_n\|_\infty = \sup_{m<x\le n} |x^{-1/3}| = m^{-1/3} \le N^{-1/3} \to 0, \qquad \text{as } N \to \infty.$$

So clearly (f_n) is Cauchy.

The case $p \in [1,\infty)$: Suppose $N \le m < n$. Then

$$\|f_m - f_n\|_p^p = \int_m^n |x^{-1/3}|^p\,dx = \int_m^n x^{-p/3}\,dx.$$

If $p > 3$, then we find that

$$\|f_m - f_n\|_p^p \le \int_N^\infty x^{-p/3}\,dx = \left[\frac{1}{1-p/3}x^{1-p/3}\right]_N^\infty = \frac{N^{1-p/3}}{p/3-1} \to 0, \qquad \text{as } N \to \infty,$$

so clearly (f_n) is Cauchy in this case. If $p \le 3$, then we easily see that for any m, we have

$$\|f_m - f_n\|_p^p \to \infty, \qquad \text{as } n \to \infty,$$

so (f_n) is not Cauchy in this case.

Answer: For $p \in (3,\infty]$.

Problem 2: (7p) Define for $n = 1, 2, 3, \dots$ the functions $f_n = \chi_{[-n,n]}$ so that

$$f_n(x) = \begin{cases} 1 & \text{when } x \in [-n,n], \\ 0 & \text{when } x \notin [-n,n]. \end{cases}$$

(a) (4p) Specify the Fourier transform $\hat{f}_n$.

(b) (3p) Consider the sequence $(\hat{f}_n)_{n=1}^{\infty}$. Specify its limit point in $\mathcal{S}^*(\mathbb{R})$.

Solution:

(a) With $\beta = 1/\sqrt{2\pi}$ we have, directly from the definition, exploiting that f_n is even,

$$\hat{f}_n(t) = \beta\int_{-\infty}^{\infty} e^{-ixt}\,f_n(x)\,dx = \beta\int_{-n}^{n}\cos(xt)\,dx = \beta\left[\frac{\sin(xt)}{t}\right]_{-n}^{n} = 2\beta\frac{\sin(nt)}{t} = \sqrt{\frac{2}{\pi}}\,\frac{\sin(nt)}{t}.$$

(b) Determining the limit of $(\hat{f}_n)$ seems a little tricky, at least to me. But: setting $f(x) = 1$, we trivially have $f_n \to f$ in $\mathcal{S}^*$. Since $\mathcal{F}:\ \mathcal{S}^* \to \mathcal{S}^*$ is continuous, it follows that $\hat{f}_n \to \hat{f}$. Now all that remains is to determine $\hat{f}$. We recall that $\mathcal{F}^*\delta = 1/\sqrt{2\pi}$. It follows that $\mathcal{F}1 = \sqrt{2\pi}\delta$.

Problem 3: (9p) Consider the function $f(x) = e^{-x^2/2}$ as a member of $L^2(\mathbb{R})$. Recall that its Fourier transform is $\hat{f}(t) = e^{-t^2/2}$.

(a) (3p) Set $g(x) = e^{-(x-1)^2/2}$. Specify $\hat{g}$.

(b) (3p) Set $h(x) = x\,e^{-x^2/2}$. Specify $\hat{h}$.

(c) (3p) Set $k(x) = e^{-x^2}$. Specify $\hat{k}$.

Solution:

(a) Use the change of variable $y = x - 1$ in the Fourier integral:

$$\hat{g}(t) = \beta\int_{-\infty}^{\infty} e^{-ixt} f(x-1)\,dx = \beta\int_{-\infty}^{\infty} e^{-i(y+1)t} f(y)\,dx = \beta e^{-it}\int_{-\infty}^{\infty} e^{-iyt} f(y)\,dx = e^{-it}\hat{f}(t) = e^{-it}e^{-t^2/2}.$$

(b) Use that the integrand and its derivative are absolutely integrable to get

$$\hat{h}(t) = \beta\int_{-\infty}^{\infty} e^{-ixt}\,x f(x)\,dx = \beta\int_{-\infty}^{\infty}\left(i\frac{d}{dt}e^{-ixt}\right) f(x)\,dx = i\frac{d}{dt}\hat{f}(t) = -it\,e^{-t^2/2}.$$

(c) Use the change of variable $y = \sqrt{2}\,x$ in the Fourier integral

$$\hat{k}(t) = \beta\int_{-\infty}^{\infty} e^{-ixt}\,f(\sqrt{2}x)\,dx = \beta\int_{-\infty}^{\infty} e^{-iyt/\sqrt{2}}\,f(y)\,dy/\sqrt{2} = \hat{f}(t/\sqrt{2})/\sqrt{2} = \frac{1}{\sqrt{2}}\,e^{-t^2/4}.$$

Note: You do not need to derive the expressions like I did here. Simply invoking the theorem in the text book is fine. If the idea of a solution was correct with just an error of a sign or a scaling constant, then 2p was awarded. (I'm not 100% sure the signs etc in the formalas above are correct...)

Problem 4: (18p) Let $p, q \in [1, \infty)$. Set $I = [0, 1]$. Circle the correct answer. No penalties for guessing (so 3p for correct answer, 0p for incorrect or no answer).

(a) (3p) If $p < q$, then it is necessarily the case that $L^p(\mathbb{R}) \subseteq L^q(\mathbb{R})$. TRUE / FALSE.

(b) (3p) If $q < p$, then it is necessarily the case that $L^p(\mathbb{R}) \subseteq L^q(\mathbb{R})$. TRUE / FALSE.

(c) (3p) If $p < q$, then it is necessarily the case that $L^p(I) \subseteq L^q(I)$. TRUE / FALSE.

(d) (3p) If $q < p$, then it is necessarily the case that $L^p(I) \subseteq L^q(I)$. TRUE / FALSE.

(e) (6p) Provide on a separate sheet motivations for two of your answers (pick any two).

Solution:

(a) FALSE. A counterexample of a function $f \in L^p(\mathbb{R}) \backslash L^q(\mathbb{R})$ is

$$f(x) = x^{-\alpha}\, \chi_{(0,1)},$$

where α is chosen so that $1/q < \alpha < 1/p$.

(b) FALSE. A counterexample of a function $f \in L^p(\mathbb{R}) \backslash L^q(\mathbb{R})$ is

$$f(x) = x^{-\alpha}\, \chi_{(1,\infty)},$$

where α is chosen so that $1/p < \alpha < 1/q$.

(c) FALSE. The counter-example in (a) works here too.

(d) TRUE. Suppose $f \in L^p(I)$ and that $q < p$. Then

$$\|f\|_{L^q(I)}^q = \int_0^1 |f(x)|^q\, dx \leq \int_0^1 \left(1 + |f(x)|^p\right) dx = 1 + \|f\|_p^p < \infty.$$

Alternatively, and with a bit more flair, invoke the Hölder inequality:

$$\|f\|_{L^q(I)}^q = \int_0^1 |f(x)|^q\, dx \leq \{\text{Hölder with parameters } n = p/q \text{ and } m = p/(p-q)\} \leq$$

$$\leq \left(\int_0^1 |f(x)|^{qn}\, dx\right)^{1/n} \left(\int_0^1 |1|^m\, dx\right)^{1/m} = \left(\int_0^1 |f(x)|^p\, dx\right)^{q/p} = \|f\|_{L^p(I)}^q.$$

Note: Providing a counter-example for some specific choices of p and q works as a way to build intuition but is not an absolutely correct answer. If you prove, say, that L^1 is not a subset of L^2, then it does not logically follow that $L^{1.5}$ is not a subset of L^2. I only deducted very minor amounts for this error, but please be aware going forwards that this is not mathematically correct.

Problem 7: (20p) Let Ω be an interval in $\mathbb{R}$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued measurable functions on Ω that converges *pointwise*. In other words, there is a function f such that

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \forall x \in \Omega.$$

Let $g \in L^2(\Omega)$, and define, whenever the integral exists,

$$\alpha_n = \int_\Omega \frac{f_n(x)}{1+|f_n(x)|}\, g(x)\, dx. \tag{1}$$

(a) (10p) Let $\Omega = [0,1]$. Prove that the integral in (1) is a well-defined Lebesgue integral that evaluates to a finite number α_n, and that

$$\lim_{n\to\infty} \alpha_n = \int_\Omega \frac{f(x)}{1+|f(x)|}\, g(x)\, dx.$$

(b) (10p) Let $\Omega = \mathbb{R}$. Provide examples of functions (f_n) and g such that (1) is well-defined as a Lebesgue integral for every n, but so that the limit of (α_n) either does not exist, or does not equal $\int_\Omega \frac{f(x)}{1+|f(x)|}\, g(x)\, dx$.

Solution:

(a) Observe that the integrand is bounded as follows:

$$\left|\frac{f_n(x)}{1+|f_n(x)|}\, g(x)\right| = \frac{|f_n(x)|}{1+|f_n(x)|}\, |g(x)| \leq |g(x)|.$$

Moreover, using the Cauchy-Schwartz inequality, we establish that

$$\int_\Omega |g(x)|\, dx \leq \left(\int_\Omega |g(x)|^2\, dx\right)^{1/2} \left(\int_\Omega 1^2\, dx\right)^{1/2} = \|g\|_{L^2(\Omega)} < \infty.$$

This means that the Lebesgue integral is well-defined, that each α_n evaluates to a finite complex number, and that the Lebesgue Dominated Convergence Theorem applies, so we can take limits under the integral

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \int_\Omega \frac{f_n(x)}{1+|f_n(x)|}\, g(x)\, dx \overset{\text{LDCT}}{=} \int_\Omega \lim_{n\to\infty} \frac{f_n(x)}{1+|f_n(x)|}\, g(x)\, dx = \int_\Omega \frac{f(x)}{1+|f(x)|}\, g(x)\, dx.$$

(b) Observe that since $m(\Omega) = \infty$, we do not know that $g \in L^1(\Omega)$ in this case. For a counter-example, use, e.g,

$$g(x) = \frac{1}{x}\, \chi_{(1,\infty)}(x), \qquad \text{and} \qquad f_n(x) = \chi_{(n,\infty)}(x).$$

Then

$$\alpha_n = \int_\Omega \frac{f_n(x)}{1+|f_n(x)|}\, g(x)\, dx = \int_n^\infty \frac{1}{1+1}\, \frac{1}{x}\, dx = \infty.$$

But the pointwise limit f is zero in this case, so

$$\int_\Omega \frac{f(x)}{1+|f(x)|}\, g(x)\, dx = \int_\Omega 0\, g(x)\, dx = 0.$$