REVIEW OF COMPACT SETS & COMPACT OPS ON H.S.

Recall that if $A \in \mathcal{B}(H)$, then TFAE:

(1) A is compact

(2) For any bounded set Ω, $A\Omega$ is pre-compact.

(3) If $(x_n)_{n=1}^{\infty}$ is a bdd seq,
then $(Ax_n)_{n=1}^{\infty}$ has a conv. subseq.

(4) If $x_n \rightharpoonup x$ weakly, then $Ax_n \to Ax$ in norm.

(5) For any $\varepsilon > 0$, $\exists$ a finite-rank op. A_ε s.t. $\|A - A_\varepsilon\| < \varepsilon$.

<u>Proof that (3) ⇒ (4)</u> Assume that (4) is false.

Then $\exists\ (x_n)$ s.t. $x_n \rightharpoonup x$, but $Ax_n \not\to Ax$.

First observe that $Ax_n \rightharpoonup Ax$. $(*)$

(Since $\forall y$: $\langle Ax_n - Ax, y\rangle = \langle x_n - x, A^* y\rangle \to 0$ since $x_n \rightharpoonup x$.)

The fact that $Ax_n \not\to Ax \Rightarrow \exists\, \varepsilon > 0$ & subseq s.t. $\|Ax - Ax_{n_j}\| \geq \varepsilon$. $(**)$

But then $(x_{n_j})_{j=1}^{\infty}$ is a bdd sequence, such that $(Ax_{n_j})_{j=1}^{\infty}$ does not have a convergent subseq.

(Note that $(*)$ means the only possible limit point for a subseq is Ax, and $(**)$ shows this is not possible.)

<u>Proof that (4) ⇒ (3)</u> Assume that (4) is true.

Let (x_n) be a bdd seq.

Banach-Alaoglu ⇒ $\exists$ weakly conv subseq
$x_{n_j} \rightharpoonup x$.

Then (4) implies that $Ax_{n_j} \to Ax$.

Recall that if Ω is a subset of a H.S.,
then TFAE:

(a) Ω is compact

(b) Every open cover of Ω has a finite subcover

(c) Every seq in Ω has a conv. subseq.

(d) Ω is closed and totally bdd
(i.e. $\forall \varepsilon > 0$, $\exists (x_j)_{j=1}^{J}$ s.t. $\Omega \subseteq \bigcup_{j=1}^{J} B_\varepsilon(x_j)$.).

We will next prove that in a H.S. H, we can add a property (e) that is analogous to property (5) for an operator.

<u>Thm</u> Let Ω be a subset of an inf. dim H.S. H.

(a) If Ω is pre-compact, and $(\varphi_n)_{n=1}^{\infty}$ is an ON-basis for H, then for any $\varepsilon > 0$, $\exists N$ s.t.

$$\|(I - P_N)x\| \leq \varepsilon \qquad \forall x \in \Omega,$$

where P_N = orthog proj^n onto span$(\varphi_1, \varphi_2, \dots, \varphi_N)$.

(b) Let Ω denote a bdd set. If there is an ON-basis $(\varphi_n)_{n=1}^{\infty}$ such that $\forall \varepsilon > 0$, $\exists N$ s.t.

$$\|(I - P_N)x\| \leq \varepsilon \qquad \forall x \in \Omega,$$

where P_N is defined as in (a), then Ω is precompact.

NOTE (corollary of (a)): Let Ω be a pre-compact set, and let $(\varphi_n)_{n=1}^{\infty}$ be any ON-set ~~in~~ in H. Then for any $\varepsilon > 0$, $\exists N < \infty$ s.t. $\sum_{n=N+1}^{\infty} |\langle \varphi_n, x \rangle|^2 \leq \varepsilon^2 \quad \forall x \in \Omega$.
To prove this apply (a) to the proj^n of H onto span$(e_n)_{n=1}^{\infty}$.

PROOF (a) Suppose that Ω is precompact.

Let $(\varphi_n)_{n=1}^{\infty}$ be an ON-basis for H.

Given $\varepsilon > 0$, pick $(x_j)_{j=1}^{J}$ s.t. $\Omega \subseteq \bigcup_{j=1}^{J} B_{\varepsilon/2}(x_j)$.

For each j, pick N_j s.t. $\|(I - P_{N_j})x_j\| < \frac{\varepsilon}{2}$.

(This is possible since $\lim_{N\to\infty} \|(I-P_N)x_j\| = 0$.)

Set $\tilde{x}_j = P_{N_j} x_j$.

Set $N = \max_{1 \le j \le J} N_j$.

Then given $x \in \Omega$, pick x_j s.t. $x \in B_{\varepsilon/2}(x_j)$. Then

$$\|(I-P_N)x\| = \|(I-P_N)(\tilde{x}_j + x - \tilde{x}_j)\| = \|\underbrace{(I-P_N)\tilde{x}_j}_{=0} + (I-P_N)(x-\tilde{x}_j)\|$$

$$= \|(I-P_N)(x-\tilde{x}_j)\| \le \|x-\tilde{x}_j\| \le \|x - x_j\| + \|x_j - \tilde{x}_j\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

(b) Suppose the cond^{ns} given in (b) hold.

Set $C' = \sup\{\|x\| : x \in \Omega\}$.

Fix $\varepsilon > 0$, and pick N s.t. $\|(I-P_N)x\| < \varepsilon/2 \quad \forall x \in \Omega$.

Set $R_N = \{x \in \text{span}(\varphi_1, \varphi_2, \dots, \varphi_N) : \|x\| \le C'\}$.

R_N is compact $\Rightarrow \exists (x_j)_{j=1}^{J}$ s.t. $R_N \subseteq \bigcup_{j=1}^{J} B_{\varepsilon/2}(x_j)$.

~~Then~~ Claim: $\bigcup_{j=1}^{J} B_{\varepsilon}(x_j)$ is an open cover of Ω.

To prove the claim, pick $x \in \Omega$.

Then $x = P_N x + (I - P_N)x$.

Since $P_N x \in R_N$, $\exists$ x_j s.t. $P_N(x) \in B_{\varepsilon/2}(x_j)$.

Then $\|x - x_j\| \le \|(P_N x - x_j) + (I-P_N)x\| \le$

$$\le \|P_N x - x_j\| + \|(I-P_N)x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

Example $H = L^2(\mathbb{T})$. Fix $C_1, C_2 < \infty$.

$\Omega = \{ f \in H : \|f\|_u \leq C_1 \text{ and } \|f'\|_u \leq C_2 \}$.

Prove that Ω is precompact!

Sol^n We use the basis $\varphi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$

$$|\langle \varphi_n, f \rangle| = \left| \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} f(x)\,dx \right| = \left| \left[\frac{e^{-inx}}{-in\sqrt{2\pi}} f(x) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in\sqrt{2\pi}} f'(x)\,dx \right| =$$

$$= \left| \frac{e^{-i\pi n}}{-in\sqrt{2\pi}} f(\pi) - \frac{e^{in\pi}}{-in\sqrt{2\pi}} f(-\pi) - \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in\sqrt{2\pi}} f'(x)\,dx \right| \leq$$

$$\leq \frac{|f(\pi)|}{n\sqrt{2\pi}} + \frac{|f(-\pi)|}{n\sqrt{2\pi}} + \frac{1}{n\sqrt{2\pi}} 2\pi \|f'\|_u \leq$$

$$\leq \frac{1}{n} \left(\frac{1}{\sqrt{2\pi}} C_1 + \frac{1}{\sqrt{2\pi}} C_1 + \frac{2\pi}{\sqrt{2\pi}} C_2 \right) =: \frac{C_3}{n}$$

Note: C_3 does <u>not</u> depend on f!

Then $\|(I - P_N) f\|^2 = \sum_{|n| > N} |\langle \varphi_n, f \rangle|^2 \leq \sum_{|n| > N} \frac{C_3^2}{n^2}$

For any $\varepsilon > 0$, we can pick N s.t. $\sum_{|n| > N} \frac{C_3^2}{n^2} < \varepsilon^2$,

so $\|(I - P_N) f\| < \varepsilon$.