## APPM4720/5720 — Homework 2

Hand in solutions to at least three of the following four problems.

**Question 2.1:** Consider three different ways of computing the DFT:

(a) Apply it directly via the formula

$$\hat{\boldsymbol{x}}(n) = [\mathsf{F}_N\,\boldsymbol{x}](n) = \sum_{m=0}^{N-1} e^{-i\,2\,\pi\,m\,n/N}\,\boldsymbol{x}(m), \qquad n = 0,\,1,\,2,\,\dots,\,N-1.$$

(It is strongly recommended to *vectorize* the matrix-vector product.)

(b) Build your own implementation of the basic FFT algorithm described in the course notes. (This is the *Cooley-Tukey* algorithm.)

(c) Apply the built-in Matlab command FFT. (Last time I checked, Matlab used the *FFTW* package, but this may have changed.)

Measure the asymptotic timing for each of the methods, and comment on what you observe.

**Question 2.2:** Solve (analytically) the Laplace equation with Dirichlet boundary data on a domain exterior to a unit disc. In other words, let $\Omega$ and $\Gamma$ be defined by

$$\Omega = \{x \in \mathbb{R}^2 :\ |x| > 1\}$$
$$\Gamma = \{x \in \mathbb{R}^2 :\ |x| = 1\},$$

and consider the boundary value problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ \quad u(x) = g(x), & x \in \Gamma. \end{cases}$$

You may assume that $g \in L^1 \cap L^2$, that

$$\int_0^{2\pi} g(\cos\theta, \sin\theta) = 0,$$

and then require the solution $u$ to decay at infinity.

**Question 2.3:** In this problem, we will numerically solve the equation

$$\begin{cases} -u''(x) = & f(x), \qquad x \in (0,\pi), \\ u(0) = & 0, \\ u(\pi) = & 0, \end{cases}$$

for a few different right hand sides. For each given $f$, compute the solution using either the basis functions $\{\sqrt{2/\pi}\,\sin(n\,x)\}_{n=1}^{\infty}$, or the basis functions $\{\sqrt{1/\pi}\,e^{2inx}\}_{n\in\mathbb{Z}}$ (with appropriate corrections for the boundary conditions). Consider the following choices of $f$:

(a) $f(x) = e^{(\sin x)^2}$.

(b) $f(x) = e^{\sqrt{x}}$.

(c) $f(x) = \cos(100\,x)$.

(d) $f(x) = \cos(100.1\,x)$.

(e) $f(x) = |x-1|^{-0.25}$.

(f) $f(x) = \begin{cases} 0 & 0 < x \leq 1 \\ 1 & 1 < x \leq 2 \\ 2 & 1 < x < \pi \end{cases}$

Your solution should contain the following: (i) A plot of the solution $u$. (ii) Plots of $|\hat{f}_n|$ and $|\hat{u}_n|$ versus $n$ for each of the functions and a discussion of the rates of decay.

**Question 2.4:** Construct a program that uses Fourier methods to solve the *heat equation*

$$\begin{cases} \frac{d^2u}{dx^2} = & \frac{du}{dt}, \qquad x \in (0,\pi), t > 0, \\ u(0,t) = & 0, \qquad t > 0 \\ u(\pi,t) = & 0, \qquad t > 0 \\ u(x,0) = & f(x), \qquad 0 < x < \pi \\ u_t(x,0) = & 0, \end{cases}$$

where $f(x) = e^{(\sin x)^2}$ or $f(x) = |x-1|^{-0.25}$. For both choices of $f$, produce plots of the solution at a few different times $t$ (your choice!). In addition, produce a plot of the function

$$U(t) = \int_0^{\pi} |u(x,t)|^2\,dx.$$

Repeat the exercise, but now consider the *wave equation*

$$\begin{cases} \frac{d^2v}{dx^2} = & \frac{d^2v}{dt^2}, \qquad x \in (0,\pi), t > 0, \\ v(0,t) = & 0, \qquad t > 0 \\ v(\pi,t) = & 0, \qquad t > 0 \\ v(x,0) = & f(x), \qquad 0 < x < \pi \\ v_t(x,0) = & 0, \end{cases}$$

Produce a plot of

$$V(t) = \int_0^{\pi} |v(x,t)|^2\,dx.$$

*Hint:* For your own amusement, you may want to create animations of the solutions using the Matlab movie command.