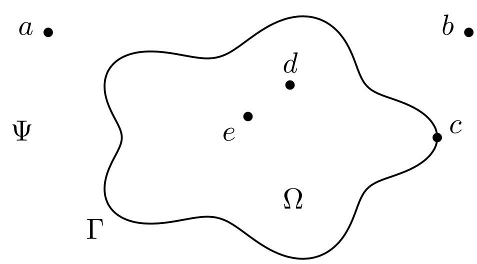
APPM4720/5720 — Homework 5

The problems in this homework rely on the geometry shown below:



The contour Γ is defined via

$$\Gamma = \{x = (G_1(t),\, G_2(t)) :\ t \in [0,\, 2\pi)\}.$$

where

$$\begin{aligned} G_1(t) =& 1.5\ \cos(t) + 0.1\ \cos(6\,t) + 0.1\ \cos(4\,t), \\ G_2(t) =& \sin(t) + 0.1\ \sin(6\,t) - 0.1\ \sin(4\,t). \end{aligned}$$

The coordinates of the points are

$$a = (-2,\, 1) \quad b = (2,\, 1) \quad c = (1.7,\, 0) \quad d = (0.3,\, 0.5) \quad e = (-0.1,\, 0.2).$$

The domain *interior* to Γ is Ω, and the domain *exterior* to Γ is Ψ.

Problem 5.1: Consider the *exterior* Neumann problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Psi, \\ u_n(x) = r(x), & x \in \Gamma, \end{cases} \tag{1}$$

where

$$f(x_1,\, x_2) = x_1\, e^{\sin(10\,x_2)},$$

and where r is defined to equal f, but shifted so that $\int_\Gamma r = 0$:

$$r(x) = f(x) - \frac{1}{|\Gamma|}\int_\Gamma f(x)dl(x).$$

Let u have the representation

$$u(x) = [S\,\sigma](x) = \int_\Gamma \frac{1}{2\pi}\,\log\frac{1}{|x - x'|}\,\sigma(x')\,dl(x').$$

Your task is to form an equation for σ, discretize this equation, solve the equation, and then to evaluate the function u. (You will find the relevant formulas in the course notes!)

Your answer should include a print-out of your Matlab code, and an accurate estimate of

$$u(a) - u(b).$$

(Observe that $u(a)$ and $u(b)$ are not uniquely determined by the Neumann problem, but their *difference* is!)

Problem 5.2: Repeat Problem 5.1, but now solve the corresponding *interior* problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ u_n(x) = r(x), & x \in \Gamma, \end{cases} \tag{2}$$

where Ω is the domain interior to Γ.

First look for a solution of the form

$$u(x) = [S\,\sigma](x) = \int_\Gamma \frac{1}{2\pi}\,\log\frac{1}{|x-x'|}\,\sigma(x')\,dl(x').$$

This will result in a linear system

$$\mathsf{A}\,\boldsymbol{\sigma} = \boldsymbol{r}$$

where A is an $N \times N$ matrix of rank $N-1$. Verify that $\boldsymbol{r} \in \mathrm{Col}(\mathsf{A})$ (the column space, or range, of A), and then construct a solution via

$$\boldsymbol{\sigma} = \mathsf{A}^\dagger\,\boldsymbol{r},$$

where $\mathsf{A}^\dagger$ is the Moore-Penrose pseudo-inverse

$$\mathsf{A}^\dagger = \mathsf{V}(:, 1:(N-1))\,\mathsf{\Sigma}(1:(N-1), 1:(N-1))^{-1}\,\mathsf{U}(:, 1:(N-1))^*$$

where

$$\mathsf{A} = \mathsf{U}\,\mathsf{\Sigma}\,\mathsf{V}^*$$

is the SVD of A.

Next look for a solution of the form

$$u(x) = [S\,\sigma](x) = \int_\Gamma \frac{1}{2\pi}\,\log\frac{1}{|x-x'|}\,\sigma(x')\,dl(x') + \frac{1}{2\pi}\left(\log\frac{1}{|x|}\right)\int_\Gamma \sigma(x')\,dl(x').$$

(For a motivation of this choice, see course notes.)

In your answer, simply specify the value of

$$u(d) - u(e).$$