APPM – 5720 Fast Algorithms for Big Data

Reference Homework Problem
HW: 4
Question: 2

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Q2. A is an $n \times n$ matrix

$A(i,j) > 0 \quad \forall\, i,j$

$$\sum_{i=1}^{n} A(i,j) = 1 \quad \forall j$$

(a) $\sum_{j=1}^{n} p(j) = 1$; p is a vector of non-negative numbers.

if $p' = Ap$; P.T $\sum_{j=1}^{n} p'(j) = 1$

Let us start with a 2×2 matrix

$$p'(j) = A \cdot p(j)$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

we know, $a + c = 1$.

$b + d = 1$

$e + f = 1$

we must prove ; $x + y = 1$

$$\therefore \quad x = ae + bf$$

$$y = ce + df$$

$$\begin{aligned} \therefore \quad x + y &= ae + bf + ce + df \\ &= e\underbrace{(a+c)}_{=1} + f\underbrace{(b+d)}_{=1} \\ &= e + f \end{aligned}$$

$\therefore\ x+y = e+f$
$= 1$

we can show the same for a 3x3 matrix and so on.

Thus, $\sum P'(j) = \sum_{j=1}^{n} \sum_{j=1}^{n} A(i,j) P(j)$

$$= \sum_{j=1}^{n} P(j) \underbrace{\sum_{j=1}^{n} A(i,j)}_{=1}$$

$$= \underbrace{\sum_{j=1}^{n} P(j)}_{=1} \cdot 1$$

$$= 1.$$

$$\therefore \sum_{j=1}^{n} P'(j) = 1$$

Hence Proved.

Q2

(b) Prove that A has an eigenvector corresponding to eigenvalue = 1.

We know that for an eigenvalue to exist, it must satisfy the following condition

$Av = \lambda v$

if $\lambda = 1$.

$Av = v$

We have to prove there exists a v which satisfies this equation.

Let us take A^T

now, A had dim i, j so, A^T will be j, i and each ~~to~~ row will sum to 1.

Further, we know that a matrix and it's transpose have same evals, hence, if the condition is satisfied for A^T, $\lambda = 1$ must be an eval of A and hence an evec must exist.

$\therefore$ $A^T v = v$

Let us take a 3×3 matrix

$$\therefore \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

we know that

$$a+b+c = 1$$
$$d+e+f = 1$$
$$g+h+i = 1$$

$$\therefore \text{let } v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore Av = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a+b+c \\ d+e+f \\ g+h+i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = v$$

we can generalize this to a matrix of any size.

if A^T is of size j, i; we need to take v of size $1, j$

$\therefore$ $v = (\underbrace{1, 1, 1, 1 - - -, 1}_{j})^T$

$\therefore$ $Av = v$

$\therefore$ $\lambda = 1$ is an eval of A^T

$\Rightarrow$ $\lambda = 1$ must be an eval of A

$\Rightarrow$ A must have an eigenvector with corresponding eval $= 1$

Hence Proved.