Question 1: There is no connection between the matrices and the vectors in the following questions. No motivation is required. (2p each)

(a) Suppose that $A$ is an $m \times n$ matrix of rank $k$. What is $\dim(\text{Col}(A))$?

(b) Suppose that there exists a non-zero vector $y$ such that $Ay = 0$. Does there exist a vector $b$ such that the equation $Ax = b$ has a unique solution?

(c) Is it true that $\lambda$ is an eigenvalue of $A$ if and only if $\det(A - \lambda I) = 0$?

(d) Is it true that all square matrices are diagonalizable?

(e) What conditions must a matrix satisfy for it to have a singular value decomposition?

(f) Complete the sentence: If $A$ and $B$ are matrices, then every row of $AB$ is a linear combination of the rows of the matrix . . . .

(g) Suppose that $q_1$ and $q_2$ are two non-zero vectors in $\mathbb{R}^n$ such that $q_1 + t q_2 \neq 0$ for all $t$. What is $\dim(\text{Span}\{q_1, q_2, q_1 + q_2\})$?

(h) Suppose that $A$ is an $m \times n$ matrix of rank $n$, that $b \in \text{Col}(A) \perp$ and that $b \neq 0$. How many solutions does the equation $Ax = b$ have?

(i) Suppose that $H$ is a subspace of $\mathbb{R}^n$ of dimension $k$, where $k < n$. What is the dimension of $H \perp$?

(j) Give a matrix $A$ such that $2x_1^2 + x_1 x_2 - x_2^2 = x^t A x$, where $x^t = [x_1 \ x_2]$.

(k) What is the largest possible singular value of an orthogonal matrix?
Question 2: Consider the equation

\[ \begin{align*}
A & = \begin{bmatrix}
1 & 1 & 2 & 1 & 0 \\
-1 & 1 & -2 & -1 & 1 \\
1 & 1 & 2 & 2 & 1 \\
1 & 5 & 2 & 2 & 3
\end{bmatrix}, & \mathbf{x} &= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}, & \mathbf{b} &= \begin{bmatrix}
2 \\
1 \\
0 \\
6
\end{bmatrix}.
\end{align*} \]

(a) Find all \( \mathbf{x} \) that solve (1). Your answer should either give an explicit formula for each \( x_i \) or indicate that it is a free variable. (10p)

(b) Give a basis for \( \text{Row}(A) \). (5p)

(c) Give a basis for \( \text{Col}(A) \). (5p)

Question 3: Consider \( \mathbb{P}^3 \), the vector space of polynomials of degree 3 or less, and its subspace

\[ H = \text{Span}\{1 + t + 2t^3, 1 + 3t - 4t^2, 2 + 2t + 4t^3, 1 + t + t^2 + 2t^3\} \]

(a) Does the vector \( \mathbf{p}(t) = 2 + 2t - t^2 + 4t^3 \) belong to \( H \)? (7p)

(b) What is the dimension of \( H \)? (3p)

Question 4: Consider the matrices

\[ U = \frac{1}{2} \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}, \]

and set

\[ A = U D U^t. \]

Note that \( U \) is orthogonal.

(a) What are the eigenvalues of \( A \)? (5p)

(b) Let \( \mathbf{u}_2 \) denote the second column of \( U \). What is \( A\mathbf{u}_2 \)? (5p)

(c) Determine an orthogonal matrix \( P \), and a diagonal matrix \( E \), such that \( A^2 + A^{-1} = P E P^t \). (5p)
Question 5: Consider the vectors
\[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \]
(note that \( \mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \)) and set \( V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \).

(a) Set \( \mathbf{q}_1 = \mathbf{v}_1/||\mathbf{v}_1|| \), \( \mathbf{q}_2 = \mathbf{v}_2/||\mathbf{v}_2|| \), and determine a vector \( \mathbf{q}_3 \) such that \( \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} \) forms an orthonormal basis for \( V \). (8p)

(b) Set \( \mathbf{A} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3], \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \), and find an upper triangular matrix \( \mathbf{R} \) such that \( \mathbf{A} = \mathbf{QR} \). (6p)

(c) Find a vector \( \mathbf{z} \in V \) such that \( ||\mathbf{b} - \mathbf{z}|| \leq ||\mathbf{b} - \mathbf{y}|| \) for all \( \mathbf{y} \in V \). (6p)

Question 6: If \( \mathbf{A} \) is an \( n \times n \) invertible matrix, and if \( \mathbf{u} \) and \( \mathbf{v} \) are \( n \times 1 \) vectors such that \( \mathbf{A} + \mathbf{u} \mathbf{v}^\intercal \) is invertible, then
\[ (\mathbf{A} + \mathbf{u} \mathbf{v}^\intercal)^{-1} = \mathbf{A}^{-1} - \beta \mathbf{A}^{-1} \mathbf{u} \mathbf{v}^\intercal \mathbf{A}^{-1}, \]
where
\[ \beta = \frac{1}{1 + \mathbf{v}^\intercal \mathbf{A}^{-1} \mathbf{u}}. \]

(a) Use formula (2) to compute the inverse of
\[ \mathbf{B} = \begin{bmatrix} 6 & 1 & 1 & 1 & 1 \\ 1 & 6 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}. \]

(7p) Hint: Set \( \mathbf{u} = \mathbf{v} = [1 1 1 1 1]^\intercal \).

(b) Prove the formula (2). (8p) Hint: Set \( \mathbf{X} = \mathbf{A}^{-1} - \beta \mathbf{A}^{-1} \mathbf{u} \mathbf{v}^\intercal \mathbf{A}^{-1} \), prove that \( (\mathbf{A} + \mathbf{u} \mathbf{v}^\intercal) \mathbf{X} \mathbf{A} = \mathbf{A} \), and solve for \( \mathbf{X} \).

Remark: As a curiosities, the formula (2) is a special case of something called the Sherman-Morrison-Woodbury formula which says that
\[ (\mathbf{A} + \mathbf{U} \mathbf{V}^\intercal)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1} \mathbf{U} (\mathbf{I} + \mathbf{V}^\intercal \mathbf{A}^{-1} \mathbf{U}) \mathbf{V}^\intercal \mathbf{A}^{-1}, \]
where \( \mathbf{A} \) is an invertible \( n \times n \) matrix, and \( \mathbf{U} \) and \( \mathbf{V} \) are \( n \times k \) matrices such that \( \mathbf{A} + \mathbf{U} \mathbf{V}^\intercal \) is invertible. The formula (3) is very useful in cases where one knows \( \mathbf{A} \) and wants to compute the inverse of a slightly perturbed version of \( \mathbf{A} \) (say only a couple of elements changed).

/// PG Martinsson, Dec 17, 2004