Math 222, Fall 2004: Final Exam

- Closed books, no notes, no calculators.
- Upper case letters denote matrices, boldface lower case letter vectors, and plain lower case letters scalars.
- The transpose of a matrix A is denoted A^{t}.
- The identity matrix is denoted I.
- A matrix U is said to be orthogonal if $U^{\mathrm{t}}U = I$.
- In questions 2 – 6, please provide brief explanations for the steps in your calculations.

Question 1: There is no connection between the matrices and the vectors in the following questions. No motivation is required. (2p each)

(a) Suppose that A is an $m \times n$ matrix of rank k. What is $\dim(\mathrm{Col}(A))$?

(b) Suppose that there exists a non-zero vector $\mathbf{y}$ such that $A\mathbf{y} = 0$. Does there exist a vector $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution?

(c) Is it true that λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$?

(d) Is it true that all square matrices are diagonalizable?

(e) What conditions must a matrix satisfy for it to have a singular value decomposition?

(f) Complete the sentence: If A and B are matrices, then every row of AB is a linear combination of the rows of the matrix ....

(g) Suppose that $\mathbf{q}_1$ and $\mathbf{q}_2$ are two non-zero vectors in $\mathbb{R}^n$ such that $\mathbf{q}_1 + t\mathbf{q}_2 \neq 0$ for all t. What is $\dim(\mathrm{Span}\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_1 + \mathbf{q}_2\})$?

(h) Suppose that A is an $m \times n$ matrix of rank n, that $\mathbf{b} \in \mathrm{Col}(A)^{\perp}$ and that $\mathbf{b} \neq 0$. How many solutions does the equation $A\mathbf{x} = \mathbf{b}$ have?

(i) Suppose that H is a subspace of $\mathbb{R}^n$ of dimension k, where $k < n$. What is the dimension of $H^{\perp}$?

(j) Give a matrix A such that $2x_1^2 + x_1x_2 - x_2^2 = \mathbf{x}^{\mathrm{t}}A\mathbf{x}$, where $\mathbf{x}^{\mathrm{t}} = [x_1\, x_2]$.

(k) What is the largest possible singular value of an orthogonal matrix?

Question 2: Consider the equation

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

where

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ -1 & 1 & -2 & -1 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 5 & 2 & 2 & 3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 6 \end{bmatrix}.$$

(a) Find all $\mathbf{x}$ that solve (1). Your answer should either give an explicit formula for each x_i or indicate that it is a free variable. (10p)

(b) Give a basis for $\mathrm{Row}(A)$. (5p)

(c) Give a basis for $\mathrm{Col}(A)$. (5p)

Question 3: Consider $\mathbb{P}^3$, the vector space of polynomials of degree 3 or less, and its subspace

$$H = \mathrm{Span}\{1 + t + 2t^3,\ 1 + 3t - 4t^2,\ 2 + 2t + 4t^3,\ 1 + t + t^2 + 2t^3\}.$$

(a) Does the vector $\mathbf{p}(t) = 2 + 2t - t^2 + 4t^3$ belong to H? (7p)

(b) What is the dimension of H? (3p)

Question 4: Consider the matrices

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and set

$$A = U\,D\,U^{\mathrm{t}}.$$

Note that U is orthogonal.

(a) What are the eigenvalues of A? (5p)

(b) Let $\mathbf{u}_2$ denote the second column of U. What is $A\mathbf{u}_2$? (5p)

(c) Determine an orthogonal matrix P, and a diagonal matrix E, such that $A^2 + A^{-1} = P\,E\,P^{\mathrm{t}}$. (5p)

Question 5: Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

(note that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$) and set

$$V = \text{Span}\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}.$$

(a) Set $\mathbf{q}_1 = \mathbf{v}_1/||\mathbf{v}_1||$, $\mathbf{q}_2 = \mathbf{v}_2/||\mathbf{v}_2||$, and determine a vector $\mathbf{q}_3$ such that $\{\mathbf{q}_1,\, \mathbf{q}_2,\, \mathbf{q}_3\}$ forms an orthonormal basis for V. (8p)

(b) Set $A = [\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3]$, $Q = [\mathbf{q}_1,\, \mathbf{q}_2,\, \mathbf{q}_3]$, and find an upper triangular matrix R such that $A = QR$. (6p)

(c) Find a vector $\mathbf{z} \in V$ such that $||\mathbf{b} - \mathbf{z}|| \leq ||\mathbf{b} - \mathbf{y}||$ for all $\mathbf{y} \in V$. (6p)

Question 6: If A is an $n \times n$ invertible matrix, and if $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ vectors such that $A + \mathbf{u}\,\mathbf{v}^{\mathrm{t}}$ is invertible, then

$$\left(A + \mathbf{u}\,\mathbf{v}^{\mathrm{t}}\right)^{-1} = A^{-1} - \beta A^{-1}\mathbf{u}\mathbf{v}^{\mathrm{t}}A^{-1}, \tag{2}$$

where

$$\beta = \frac{1}{1 + \mathbf{v}^{\mathrm{t}}A^{-1}\mathbf{u}}.$$

(a) Use formula (2) to compute the inverse of

$$B = \begin{bmatrix} 6 & 1 & 1 & 1 & 1 \\ 1 & 6 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}.$$

(7p) **Hint:** Set $\mathbf{u} = \mathbf{v} = [1\,1\,1\,1\,1]^{\mathrm{t}}$.

(b) Prove the formula (2). (8p) **Hint**: Set $X = A^{-1} - \beta A^{-1}\mathbf{u}\mathbf{v}^{\mathrm{t}}A^{-1}$, prove that $\left(A + \mathbf{u}\,\mathbf{v}^{\mathrm{t}}\right) XA = A$, and solve for X.

Remark: As a curiousity, the formula (2) is a special case of something called the Sherman-Morrison-Woodbury formula which says that

$$\left(A + UV^{\mathrm{t}}\right)^{-1} = A^{-1} - A^{-1}U(I + V^{\mathrm{t}}A^{-1}U)V^{\mathrm{t}}A^{-1}, \tag{3}$$

where A is an invertible $n \times n$ matrix, and U and V are $n \times k$ matrices such that $A + UV^{\mathrm{t}}$ is invertible. The formula (3) is very useful in cases where one knows A and wants to compute the inverse of a slightly perturbed version of A (say only a couple of elements changed).

/// PG Martinsson, Dec 17, 2004