Math 222, Fall 2004, Linear algebra w/ applications.
Solutions to the final.

Question 1 For each of the subquestions, the minimum required answer is underlined.

(a) <u>k</u>

(b) <u>No</u> (If x is a soln, then so is $x+y$)

(c) <u>Yes</u>

(d) <u>No</u> (all square and symmetric matrices are, however)

(e) <u>None</u> (every matrix has an SVD)

(f) <u>B</u>

(g) <u>2</u>

(h) <u>None</u> (since b does not belong to Col(A).)

(i) <u>n-k</u>

(j) $A = \begin{bmatrix} 2 & 1/2 \\ 1/2 & -1 \end{bmatrix}$

(k) <u>1</u> (Infact, every singular value is one!)

Question 2 (a) The eq^n $A\mathbf{x}=\mathbf{b}$ has the extended system matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 2 \\ -1 & 1 & -2 & -1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 0 \\ 1 & 5 & 2 & 2 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 4 & 0 & 1 & 3 & 4 \end{array}\right] \sim$$

Scale

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -3/2 & 5/2 \\ 0 & 1 & 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We can now read the sol^n off:

$$\begin{cases} x_1 = \frac{5}{2} - 2x_3 + \frac{3}{2}x_5 \\ x_2 = \frac{3}{2} - \frac{1}{2}x_5 \\ x_3 \text{ is free} \\ x_4 = -2 - x_5 \\ x_5 \text{ is free} \end{cases}$$

(b) Since row operations like the ones used above leave Row (A) invariant, we find that

$$\text{Row}(A) = \text{Span}\underbrace{\left\{ \left[1,0,2,0,-\tfrac{3}{2}\right], \left[0,1,0,0,\tfrac{1}{2}\right], \left[0,0,0,1,1\right] \right\}}_{\text{The basis!}}$$

(c) Columns 1, 2 and 4 are pivot columns.

Thus $$\text{Col}(A) = \text{Span}\underbrace{\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix} \right\}}_{\text{The basis!}}$$

Question 3 (a) We use that $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$ using the standard basis $\mathcal{B}=\{1, t, t^2, t^3\}$. Thus $\mathbb{p} \in H$ if and only if the equation below has a solution:

$$\underset{\uparrow\ [1+t+2t^3]_{\mathcal{B}} \text{ etc}}{\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 0 & -4 & 0 & 1 \\ 2 & 0 & 4 & 2 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \underset{\uparrow\ [\mathbb{p}]_{\mathcal{B}}}{\begin{bmatrix} 2 \\ 2 \\ -1 \\ 4 \end{bmatrix}} \qquad (*)$$

We perform Gaussian elimination:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 1 & 2 \\ 0 & -4 & 0 & 1 & -1 \\ 2 & 0 & 4 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is no pivot element in the rightmost column the equation $(*)$ does have a sol^n (e.g. $c_1 = 3$, $c_2 = 0$, $c_3 = 0$, $c_4 = -1$)

Answer: Yes

(b) The dimension is <u>three</u>.

(Since the matrix in the eq^n $(*)$ has rank 3.)

Question 4 (a) $\lambda = 2, 3, -1$

(b) u_2 is an eigenvector of A with eigenvalue 3.

Thus $A u_2 = 3 u_2 = \begin{bmatrix} 3/2 \\ -3/2 \\ 3/2 \\ -3/2 \end{bmatrix}$

(c) $A^2 + A^{-1} = UDU^tUDU^t + (U^t)^{-1}D^{-1}U^{-1} = UD^2U^t + UD^{-1}U^t =$

$= U \underbrace{(D^2 + D^{-1})}_{=E} U^t$

Answer $E = D^2 + D^{-1} = \begin{bmatrix} 9/2 & 0 & 0 & 0 \\ 0 & 28/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$P = U$

Question 5 (a) First we compute a vector $\mathbb{u}_3$ so that $\{\mathbb{v}_1, \mathbb{v}_2, \mathbb{u}_3\}$ forms an orthogonal (non-normalized) basis for $\mathbb{V}$.

$$\mathbb{u}_3 = \mathbb{v}_3 - \text{proj}_{\mathbb{v}_1}\mathbb{v}_3 - \text{proj}_{\mathbb{v}_2}\mathbb{v}_3 = \mathbb{v}_3 - \frac{\mathbb{v}_3\cdot\mathbb{v}_1}{\|\mathbb{v}_1\|^2}\mathbb{v}_1 - \frac{\mathbb{v}_3\cdot\mathbb{v}_2}{\|\mathbb{v}_2\|^2}\mathbb{v}_2 =$$

$$= \begin{bmatrix} 0\\0\\2\\6 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1\\0\\1\\0 \end{bmatrix} - \frac{-14}{7}\begin{bmatrix} 1\\1\\-1\\-2 \end{bmatrix} = \begin{bmatrix} 1\\2\\-1\\2 \end{bmatrix}$$

Then $\quad \mathbb{q}_1 = \frac{\mathbb{v}_1}{\|\mathbb{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1\\0\\1\\0 \end{bmatrix} \quad \mathbb{q}_2 = \frac{1}{\sqrt{7}}\begin{bmatrix} 1\\1\\-1\\-2 \end{bmatrix} \quad \mathbb{q}_3 = \frac{\mathbb{u}_3}{\|\mathbb{u}_3\|} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1\\2\\-1\\2 \end{bmatrix}$

(b) If $A = QR$ where Q is orthogonal, then $R = Q^tA$.

$$R = Q^tA = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{7} & 1/\sqrt{7} & -1/\sqrt{7} & -2/\sqrt{7} \\ 1/\sqrt{10} & 2/\sqrt{10} & -1/\sqrt{10} & 2/\sqrt{10} \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & -2 & 6 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{7} & -2\sqrt{7} \\ 0 & 0 & \sqrt{10} \end{bmatrix}$$

(c) $\quad \mathbb{z} = \text{proj}_{\mathbb{V}}\,\mathbb{b} \underset{\uparrow}{=} QQ^t\mathbb{b} = \mathbb{q}_1(\mathbb{q}_1\cdot\mathbb{b}) + \mathbb{q}_2(\mathbb{q}_2\cdot\mathbb{b}) + \mathbb{q}_3(\mathbb{q}_3\cdot\mathbb{b}) =$

Since the columns of Q form an ON-basis for $\mathbb{V}$

$$= \frac{1}{2}\begin{bmatrix} 1\\0\\1\\0 \end{bmatrix}2 + \frac{1}{7}\begin{bmatrix} 1\\1\\-1\\-2 \end{bmatrix}(-1) + \frac{1}{10}\begin{bmatrix} 1\\2\\-1\\2 \end{bmatrix}4 = \begin{bmatrix} 1\\0\\1\\0 \end{bmatrix} + \begin{bmatrix} -1/7\\-1/7\\1/7\\2/7 \end{bmatrix} + \begin{bmatrix} 2/5\\4/5\\-2/10\\4/5 \end{bmatrix}$$

Answer: $\mathbb{z} = \begin{bmatrix} 1 - \frac{1}{7} + \frac{2}{5} \\ -\frac{1}{7} + \frac{4}{5} \\ 1 + \frac{1}{7} - \frac{2}{10} \\ \frac{2}{7} + \frac{4}{5} \end{bmatrix}$

Remark: It is also possible to solve this problem by setting $\mathbb{z} = A\hat{\mathbb{x}}$ where $\hat{\mathbb{x}}$ is the least squares sol^{n} of $A\mathbb{x} = \mathbb{b}$, in other words, $A^tA\hat{\mathbb{x}} = A^t\mathbb{b} \Rightarrow \mathbb{z} = A(A^tA)^{-1}A^t\mathbb{b}$

Question 6 (a) $B = A + u u^t$ where $A = 5I$ and $u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Thus $B^{-1} = A^{-1} - \beta A^{-1} u u^t A^{-1}$ where $\beta = \frac{1}{1 + u^t A^{-1} u} = \frac{1}{1 + \frac{1}{5} u^t u} = \frac{1}{1 + \frac{1}{5} 5} = \frac{1}{2}$

so $B^{-1} = \frac{1}{5} I - \frac{1}{2} \frac{1}{5} I u u^t \frac{1}{5} I = \frac{1}{5} I - \frac{1}{50} u u^t =$

$$= \begin{bmatrix} 1/5 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 & 0 \\ 0 & 0 & 1/5 & 0 & 0 \\ 0 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 1/5 \end{bmatrix} - \frac{1}{50} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Answer $B^{-1} = \begin{bmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{bmatrix}$ where $a = \frac{1}{5} - \frac{1}{50}$, $b = -\frac{1}{50}$

(b) We set $X = A^{-1} - \beta A^{-1} u v^t A^{-1}$ and compute the product

$(A + u v^t) X A = (A + u v^t)(I - \beta A^{-1} u v^t) =$

$= A - \beta u v^t + u v^t - \beta u v^t A^{-1} u v^t = A + u(-\beta + 1 - \beta v^t A^{-1} u) v^t =$

$= A + u \beta \Big(-1 + \underbrace{1 + v^t A^{-1} u}_{= \frac{1}{\beta}} - v^t A^{-1} u\Big) v^t = A$

Thus $(A + u v^t) X A = A \Rightarrow (A + u v^t) X = I \Rightarrow (A + u v^t)^{-1} = X$