

Math 222, Fall 2004, Linear algebra w/ applications.
Solutions to the final.

Question 1 For each of the subquestions, the minimum required answer is underlined.

(a) k

(b) No (If x is a solⁿ, then so is $x+y$)

(c) Yes

(d) No (all square and symmetric matrices are, however)

(e) None (every matrix has an SVD)

(f) B

(g) 2

(h) None (since b does not belong to $\text{col}(A)$.)

(i) $n-k$

(j) $A = \begin{bmatrix} 2 & 1/2 \\ 1/2 & -1 \end{bmatrix}$

(k) 1 (In fact, every singular value is one!)

Question 2 (a) The eqⁿ $Ax=b$ has the extended system matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array} \right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 4 & 0 & 1 & 1 & -2 \end{array} \right] \xrightarrow{-2} \sim$$

$$\xrightarrow{\text{Scale}} \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array} \right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{-1} \sim$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -3/2 & 5/2 \\ 0 & 1 & 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

We can now read the solⁿ off:

$$\begin{cases} x_1 = \frac{5}{2} - 2x_3 + \frac{3}{2}x_5 \\ x_2 = \frac{3}{2} - \frac{1}{2}x_5 \\ x_3 \text{ is free} \\ x_4 = -2 - x_5 \\ x_5 \text{ is free} \end{cases}$$

(b) Since row operations like the ones used above leave $\text{Row}(A)$ invariant, we find that

$$\text{Row}(A) = \text{Span} \left\{ \left[1, 0, 2, 0, -\frac{3}{2} \right], \left[0, 1, 0, 0, \frac{1}{2} \right], \left[0, 0, 0, 1, 1 \right] \right\}$$

The basis!

(c) Columns 1, 2 and 4 are pivot columns.

$$\text{Thus } \text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

The basis!

Question 3 (a) We use that \mathbb{P}_3 is isomorphic to \mathbb{R}^4 using the standard basis $\mathcal{B} = \{1, t, t^2, t^3\}$. Thus $p \in H$ if and only if the equation below has a solution:

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 0 & -4 & 0 & 1 \\ 2 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 4 \end{bmatrix} \quad (*)$$

\uparrow $[1+t+2t^3]_{\mathcal{B}}$ etc

\uparrow $[P]_{\mathcal{B}}$

We perform Gaussian elimination:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 1 & 2 \\ 0 & -4 & 0 & 1 & -1 \\ 2 & 0 & 4 & 2 & 4 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Since there is no pivot element in the rightmost column the equation (*) does have a solⁿ (e.g.

$$\begin{array}{l} c_1 = 3 \\ c_2 = 0 \\ c_3 = 0 \\ c_4 = -1 \end{array})$$

Answer: Yes

(b) The dimension is three.

(Since the matrix in the eqⁿ (*) has rank 3.)

Question 4 (c) $\lambda = 2, 3, -1$

(b) u_2 is an eigenvector of A with eigenvalue 3.

$$\text{Thus } Au_2 = 3u_2 = \begin{bmatrix} 3/2 \\ -3/2 \\ 3/2 \\ -3/2 \end{bmatrix}$$

$$\begin{aligned} \text{(c) } A^2 + A^{-1} &= UDU^t UDU^t + (U^t)^{-1} D^{-1} U^{-1} = UD^2U^t + UD^{-1}U^t = \\ &= U \underbrace{(D^2 + D^{-1})}_{= E} U^t \end{aligned}$$

Answer $E = D^2 + D^{-1} = \begin{bmatrix} 9/2 & 0 & 0 & 0 \\ 0 & 28/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$P = U$$

Question 5 (a) First we compute a vector u_3 so that $\{v_1, v_2, u_3\}$ forms an orthogonal (non-normalized) basis for V .

$$u_3 = v_3 - \text{proj}_{v_1} v_3 - \text{proj}_{v_2} v_3 = v_3 - \frac{v_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{v_3 \cdot v_2}{\|v_2\|^2} v_2 =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{-14}{7} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

Then $q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $q_2 = \frac{1}{\sqrt{7}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix}$ $q_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$

(b) If $A = QR$ where Q is orthogonal, then $R = Q^+ A$.

$$R = Q^+ A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & -2/\sqrt{2} \\ 1/\sqrt{10} & 2/\sqrt{10} & -1/\sqrt{10} & 2/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & -2 & 6 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{7} & -2\sqrt{7} \\ 0 & 0 & \sqrt{10} \end{bmatrix}$$

(c) $z = \text{proj}_V b = Q Q^+ b = q_1 (q_1 \cdot b) + q_2 (q_2 \cdot b) + q_3 (q_3 \cdot b) =$

Since the columns of Q form an ON-basis for V

$$= \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} 2 + \frac{1}{7} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix} (-1) + \frac{1}{10} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} 4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/7 \\ -1/7 \\ 1/7 \\ 2/7 \end{bmatrix} + \begin{bmatrix} 2/5 \\ 4/5 \\ -2/10 \\ 4/5 \end{bmatrix}$$

Answer: $z = \begin{bmatrix} 1 - \frac{1}{7} + \frac{2}{5} \\ -\frac{1}{7} + \frac{4}{5} \\ 1 + \frac{1}{7} - \frac{2}{10} \\ \frac{2}{7} + \frac{4}{5} \end{bmatrix}$

Remark: It is also possible to solve this problem by setting $z = A \hat{x}$ where \hat{x} is the least squares solution of $Ax = b$, in other words, $A^+ A \hat{x} = A^+ b \Rightarrow z = A(A^+ A)^{-1} A^+ b$

