Math 222a: First midterm exam

Time: 7.00pm - 8.15pm, Oct 5, 2004.

Closed books, no notes, no calculators.

Question 1: Consider the equation

$$Ax = b \tag{1}$$

where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$$

(a) Construct all solutions of equation (1). (Your answer should for each x_i either give a formula or indicate that it is a free variable.)
(b) Does the equation $A\boldsymbol{x} = \boldsymbol{c}$ have a solution for every $\boldsymbol{c} \in \mathbb{R}^3$?
(c) Does there exist a $\boldsymbol{c} \in \mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \boldsymbol{c}$ has a unique solution?

Question 2: Let A be a 4×3 matrix that through row operations can be transformed to the matrix

$$A' = \begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

where a cross denotes an arbitrary number.
(a) Is that map $\boldsymbol{x} \mapsto A\boldsymbol{x}$ onto?
(b) Are the columns of A linearly independent?
(c) Does there exist a $\boldsymbol{c} \in \mathbb{R}^4$ such that the equation $A\boldsymbol{x} = \boldsymbol{c}$ has a unique solution?

Question 3: Suppose that

$$Z = ABCDE.$$

where A, B, C, D, E and Z are square matrices of the same size. Suppose further that A, B, D, E and Z are invertible. Prove that C is invertible and give a formula for C^{-1} (in terms of A, B, D, E, Z and their inverses).

Question 4: Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -1 \\ -2 & -2 & 4 \end{bmatrix}.$$

(a) Construct A^{-1}.

(b) Use your result from (a) to solve the following system of equations:

$$\begin{cases} x_1 & +2x_2 & -x_3 & = 0, \\ 2x_1 & +4x_2 & -x_3 & = 1, \\ -2x_1 & -2x_2 & +4x_3 & = 0. \end{cases}$$

(c) Use your result from (a) to solve the following system of equations:

$$\begin{cases} x_1 & +2x_2 & -2x_3 & = 0, \\ 2x_1 & +4x_2 & -2x_3 & = 1, \\ -x_1 & -x_2 & +4x_3 & = 0. \end{cases}$$

Question 5: Suppose that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and that} \quad \boldsymbol{v}_4 = \begin{bmatrix} 1 \\ t \\ 1 \\ 2 \end{bmatrix}.$$

For what values of t does the set $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ span $\mathbb{R}^4$?

Question 6: Suppose that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is a linearly independent set of m-dimensional column vectors, that the matrix A is given by

$$A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3],$$

and that the equation

$$A\boldsymbol{x} = \boldsymbol{b},$$

does not have a solution.

(a) Prove that the set $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{b}\}$ is linearly independent.

(b) What can you say about m?