Solutions: Midterm 1, Math 222a, Oct 5, 2004

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(1) (a) We form the augmented coefficient matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & -1 & 0 & -1 \\ 1 & 2 & 1 & 3 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 is free $\quad$ x_4 is free

Answer (a) $\begin{cases} x_1 = 1 - 2x_2 - x_4 \\ x_2 \text{ is free} \\ x_3 = 3 - 2x_4 \\ x_4 \text{ is free} \end{cases}$

(b) No, since there are fewer pivot elements than there are rows.

(c) No, since there are free variables.

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(2) (a) No, since there are fewer pivot elements than there are rows.

(b) Yes, since there are equally many pivot elements and columns.

(c) Yes, since there are no free variables.

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(3) First we derive a formula for C:

$$Z = ABCDE \Rightarrow A^{-1}Z = BCDE \Rightarrow B^{-1}A^{-1}Z = CDE \Rightarrow B^{-1}A^{-1}ZE^{-1} = CD \Rightarrow B^{-1}A^{-1}ZE^{-1}D^{-1} = C$$

Since $C = B^{-1}A^{-1}ZE^{-1}D^{-1}$, it is a product of invertible matrices (note that $B^{-1}, A^{-1}, E^{-1}, D^{-1}$ are all invertible since B, A, E, D are). Thus, it must itself be invertible.

To find C^{-1} we repeatedly use the formula $(XY)^{-1} = Y^{-1}X^{-1}$:

$$C^{-1} = (B^{-1}A^{-1}ZE^{-1}D^{-1})^{-1} = D(B^{-1}A^{-1}ZE^{-1})^{-1} = DE(B^{-1}A^{-1}Z)^{-1} =$$

$$= DEZ^{-1}(B^{-1}A^{-1})^{-1} = \underline{\underline{DEZ^{-1}AB}}$$

(4) (a)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 3 & -1 & 1/2 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 3 & -1 \\ 0 & 1 & 0 & 3 & -1 & 1/2 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

<u>Answer</u> $A^{-1} = \begin{bmatrix} -7 & 3 & -1 \\ 3 & -1 & 1/2 \\ -2 & 1 & 0 \end{bmatrix}$

(b) The coefficient matrix is A:

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \text{middle column of } A^{-1} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

(c) The coefficient matrix is the <u>transpose</u> of A:

$$A^t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (A^t)^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (A^{-1})^t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 & 3 & 2 \\ 3 & -1 & 1 \\ -1 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1/2 \end{bmatrix}$$

Note that this is the middle <u>row</u> of A^{-1}!

(5) Form the matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$.

We use that

S spans $\mathbb{R}^4 \Leftrightarrow A\mathbf{x}=\mathbf{b}$ has a sol^n for every $\mathbf{b} \Leftrightarrow A$ has four pivots.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & t \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & t-1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & t-1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3-t \end{bmatrix}$$

We see that there are four pivots if and only if $t \neq 3$

Answer: $t \neq 3$

(6) Suppose that the numbers x_1, x_2, x_3, x_4 satisfy the eq^n

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 + \mathbf{b} x_4 = 0. \qquad (*)$$

The number x_4 must be zero since otherwise $\mathbf{x} = \begin{bmatrix} -x_1/x_4 \\ -x_2/x_4 \\ -x_3/x_4 \end{bmatrix}$ would solve the equation $A\mathbf{x} = \mathbf{b}$.

Thus eq^n $(*)$ simplifies to read

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{0}. \qquad (**)$$

Since $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a linearly independent set, eq^n $(**)$ implies that $x_1 = x_2 = x_3 = 0$.

We have shown that eq^n $(*)$ can only be true if $x_1 = x_2 = x_3 = x_4 = 0$.

This shows that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}\}$ is a linearly indep. set.