Group 1 (left wing)
Solutions

I. (a) We are looking at a typical nonlinear equation of the form $\dot{x} = f(x)$, where in our case $f(x) = k_1ax - k_{-1}x^2$. To find its fixed points, we set $f(x) = 0$:

$$k_1ax - k_{-1}x^2 = 0 \Leftrightarrow x(k_1a - k_{-1}) = 0 \Leftrightarrow x_1 = 0 \text{ or } x_2 = \tfrac{k_1a}{k_{-1}}$$

To establish their stability, we look at the sign of the derivative of f at the respective fixed point (I will do it analytically, but you can also see this by graphing f and interpreting the parabola):

$$f'(x) = k_1a - 2k_{-1}x$$

$f'(0) = k_1a > 0$, hence x_1 is a repeller

$f'(\frac{k_1a}{k_{-1}}) = -k_1a$, so x_2 is an attractor

Both fixed points have the same characteristic time scale: $\frac{1}{k_1a}$.

(c) We can separate the equation and integrate both sides:

$$\frac{dx}{k_1ax - k_{-1}x^2} = dt$$

To calculate the integral of the left-hand side, we use partial fractions. For a reminder on how to use this method, take a look at any Calculus II course on integration).

We will be looking for constants A and B (which of course may depend on the parameters a, k_1 and k_{-1}) such that:

$$\frac{1}{x(k_1a - k_{-1}x)} = \frac{A}{x} + \frac{B}{k_1a - k_{-1}x}$$

We cross-multiply and eliminate the denominator to obtain:

$$A(k_1a - k_{-1}x) + Bx = 1, \ \forall x$$

Make $x = 0$ and $x = \frac{k_1a}{k_{-1}}$ in the above, and get that

$A = \dfrac{1}{k_1a}$ and $B = \dfrac{k_{-1}}{k_1a}$.

Our left-side integral splits up then as:

$$\frac{1}{k_1 a}\int \frac{1}{x}dx + \frac{k_{-1}}{k_1 a}\int \frac{1}{k_1 a - k_{-1}x}dx$$

$$= \frac{1}{k_1 a}\ \ln|x| + \frac{k_{-1}}{k_1 a}\frac{1}{-k_{-1}}\ \ln|k_1 a - k_{-1}x|$$

$$= \frac{1}{k_1 a}\ln|\frac{x}{k_1 a - k_{-1}x}|$$

Our equation becomes then:

$$\frac{1}{k_1 a}\ln|\frac{x}{k_1 a - k_{-1}x}| = t + C$$

where C is an arbitrary constant given by the initial condition $x(0) = x_0$:

$$C = \frac{1}{k_1 a}\ln|\frac{x_0}{k_1 a - k_{-1}x_0}|$$

So the general solution (given implicitly) is:

$$\frac{1}{k_1 a}\ln\left|\frac{x}{k_1 a - k_{-1}x}\frac{k_1 a - k_{-1}x_0}{x_0}\right| = t$$

Explicitly, this will be:

$$\frac{x(k_1 a - k_{-1}x_0)}{x_0(k_1 a - k_{-1}x)} = e^{ak_1 t}$$

Solving for x, you get the explicit formula with respect to t:

$$x(t) = \frac{k_1 a x_0 e^{k_1 a t}}{k_1 a + x_0 k_{-1}(e^{k_1 a t} - 1)}$$

II. (a) We write the equation as:

$$\frac{KdN}{rN(K-N)} = dt$$

and again integrate using partial fractions (remember, the “argument” is N):

$$\frac{1}{N(K-N)} = \frac{1}{K}(\frac{1}{N} + \frac{1}{K-N})$$

The left side becomes:

$$\frac{K}{r}\frac{1}{K}(\int \frac{1}{N}dN + \int \frac{1}{K-N}dN) = \frac{1}{r}\ln|\frac{N}{K-N}|$$

Hence the equation gives us that:

$$\frac{1}{r}\ln\left|\frac{N}{K-N}\right| = t + C$$

where $C = \frac{1}{r}\ln\left|\frac{N_0}{K-N_0}\right|$

We write N explicitly as a function of t just like in model I. If we choose an initial condition $N_0 < K$, we get:

$$N(t) = \frac{N_0 K e^{tr}}{K + N_0(e^{tr} - 1)}$$

(b) We can rewrite the equation in the standard form:$\dot{N} = f(N)$ as follows:

$$\dot{N} = N(r - a(N-b)^2)$$

We assume all values of the parameters a, b and r to be positive, to be consistent with their physical significance.
The fixed points are $N_1 = 0$, $N_2 = b - \sqrt{\frac{r}{a}}$ and $N_3 = b + \sqrt{\frac{r}{a}}$
Using the product rule:

$$f'(N) = r - a(N-b)^2 + N(-2a(N-b))$$

$$f'(0) = r - ab^2$$

$$f'(N_2) = f'\left(b - \sqrt{\frac{r}{a}}\right) = r - a(N_2 - b)^2 - 2aN_2(N_2 - b)$$
$$= -2aN_2(N_2 - b) = -2a(N_2 - b)^2 - 2ab(N_2 - b) = -2r - 2ab(N_2 - b) = 2\sqrt{ar}\left(b - \sqrt{\frac{r}{a}}\right)$$

$$f'(N_3) = f'\left(b + \sqrt{\frac{r}{a}}\right) = -2r - 2ab(N_3 - b) = -2\sqrt{ar}\left(b + \sqrt{\frac{r}{a}}\right) < 0$$

In conclusion: The system has three fixed points, if $r < ab^2$. Then $x = 0$ and $x = N_3$ are attractors, and $x = N_2 < N_3$ is repelling. If $r > ab^2$, then $N_2 < 0$, so the system has only two fixed points. $x = 0$ is the only repeller, and $x = N_3$ is attracting.