Homework 8

6.3.10 (a) The Jacobian at the fixed point $(x^*, y^*) = (0, 0)$ has trace $\tau = -1$ and determinant $\Delta = 0$, so for the corresponding linear system $(0, 0)$ is a non-isolated fixed point (there is a whole line of them, actually, as one of the eigenvalues is zero).
(b) We know that, in this situation, the information we get from the linearization might be deceiving, and so it is: When we set $(\dot{x}, \dot{y}) = (0, 0)$ to find the fixed points and solve, we get $(x, y) = (0, 0)$. As $(0, 0)$ is the only fixed point, it will automatically be isolated (there are no others :)
(c) Plot the nullclines (the two axes for $\dot{x} = 0$ and $y = x^2$ for $\dot{y} = 0$) and sketch the vector field along them. The asrrows are pointing into the origin in some direction (along the y-axis, for instance) and away from the origin in other directions (along the nullcline $y = x^2$). So the orrigin is a saddle point.

6.3.14

$$\begin{cases} \dot{x} = & -y + ax^3 \\ \dot{y} = & x + ay^3 \end{cases}$$

The linearization hints that the origin is a center, but we learnt not to trust that. So we want to check the stability directly, which does not necessarily mean we have to solve the system altogether. Please, note the similarity with example 6.3.2:

$$\begin{cases} \dot{x} = & -y + ax(x^2 + y^2) \\ \dot{y} = & x + ay(x^2 + y^2) \end{cases}$$

What we did in that case was to write (x, y) in polar coordinates as $(x, y) = (r \cos\theta, r \sin\theta)$. We try the same thing for our system, which does not have the symmetry of the one in the example, so looks impossible to solve. However, start with the relation: $r^2 = x^2 + y^2$. If we differentiate, we get:

$$r\dot{r} = x\dot{x} + y\dot{y} = x(-y + ax^3) + y(x + ay^3) = a(x^4 + y^4)$$

For $a > 0$ this implies that $\dot{r} > 0 \; \forall \, (x, y) \neq (0, 0)$, which makes the origin a repeller. For $a < 0$ the origin is an attractor and for $a = 0$ it is a center.

6.5.1(a) The equilibrium points are given by $x^3 - x = 0$: $x = 0, x = \pm 1$.

(b) $\ddot{x} = x^3 - x$ could be rewritten as:

$\dot{x}\ddot{x} = \dot{x}x^3 - \dot{x}x$, so:

$\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = \frac{d}{dx}(\frac{x^4}{4} - \frac{x^2}{2})$, hence:

$E(x) = \dot{x}^2 - \frac{x^4}{2} + x^2$ is conserved along trajectories (is an energy function for the system).

6.5.2 (a) The fixed points of $\ddot{x} = x - x^2$ are $x = 0$ and $x = 1$.

(b) The corresponding 2D nonlinear system is:

$$\begin{cases} \dot{x} = & y \\ \dot{y} = & x - x^2 \end{cases}$$

Studying the linear stability we discover that $x = 0$ is a saddle point ($\tau = 0$ and $\Delta = -1$) and $x = 1$ is something else (a center or a spiral, but certainly not a saddle)($\tau = 0$ and $\Delta = 1$). So our homoclinic trajectory "come out" and "go into" $(0, 0)$.

Calculate an energy function for the system:

$E(x) = (\dot{x})^2 - [x^2 - \frac{2}{3}x^3]$

E is constant along trajectories, so may obtain their equations by setting $E = k$, for any value of the constant k:

$y^2 = x^2 - \dfrac{2}{3}x^3 + k$,

Imposing the conditions that $(x, y) \to (0, 0)$ and $\frac{y}{x} \to \pm 1$ as $t \to \infty$, we get $k = 0$. In other words, our equation is:

$$y^2 = x^2 - \frac{2}{3}x^3$$

Our homoclinic trajectory will also be of the form above.

We look at the stability of the saddle point $(0, 0)$. The Jacobian has two eigenvalues: $\lambda_1 = 1$ (with corresponding unstable direction $v_1 = (1, 1)$) and $\lambda_2 = -1$ (with corresponding stable direction $v_2 = (1, -1)$. Our homoclinic trajectory will leave in the unstable direction and come back along the stable direction. In other

7.2.9 We want to check weather there exists a function $V(x, y)$ such that $\dot{x} = \frac{dV}{dx}$ and $\dot{y} = \frac{dV}{dy}$. If such a function exists, then is must satisfy $\frac{d^2V}{dxdy} = \frac{d^2V}{dydx}$. In both cases, we first check this condition, and only if it is satisfied, we keep searching for the formula of V, otherwise we conclude that the system is not a gradient system.

(a) $\begin{cases} \frac{dV}{dx} = \dot{x} & y + x^2 y \\ \frac{dV}{dy} = \dot{y} & -x + 2xy \end{cases}$

$\dfrac{d^2V}{dxdy} = \dfrac{d}{dx}(-x + 2xy) = -1 + 2y$

$\dfrac{d^2V}{dydx} = \dfrac{d}{dy}(y + x^2 y) = 1 + x^2$

Hence the system is not a gradient system.

(b) $\begin{cases} \frac{dV}{dx} = \dot{x} & -2xe^{x^2+y^2} \\ \frac{dV}{dy} = \dot{y} & -2ye^{x^2+y^2} \end{cases}$

$\dfrac{d^2V}{dxdy} = \dfrac{d}{dx}(-2ye^{x^2+y^2}) = -4xye^{x^2+y^2}$

$\dfrac{d^2V}{dydx} = \dfrac{d}{dy}(-2xe^{x^2+y^2}) = -4xye^{x^2+y^2}$

So we continue searching for V. We start with $\frac{dV}{dx} = -2xe^{x^2+y^2}$ and integrate:

$$V(x, y) = \int -2xe^{x^2+y^2} dx = -e^{x^2+y^2} + C(y),$$

where $C(y)$ is a "constant" that does not depend on the variable x, but may depend on y.

We then use that $\frac{dV}{dy} = -2ye^{x^2+y^2}$, which now translates as:

$$\frac{d}{dy}(-e^{x^2+y^2} + C(y)) = -2ye^{x^2+y^2}$$

hence $C'(y) = 0$, so $C(y)$ really is a constant: $C(y) = constant = C$.

In conclusion:

$$V(x, y) = -e^{x^2+y^2} + C$$

7.2.12 We are looking for a Lyapunov function of the form $V(x, y) = x^m + ay^n$. We calculate:

$$\dot{V} = mx^{m-1}\dot{x} + any^{n-1}\dot{y} = mx^{m-1}(-x+2y^3-2y^4) + any^{n-1}(-x-y-xy)$$

We are looking for convenient values of a, m and n that make $V(x, y) > 0$, $\forall (x, y) \neq (0, 0)$, $V(0, 0) = 0$ and $\dot{V} < 0$, $\forall (x, y) \neq (0, 0)$.

$$\dot{V} = -mx^m + 2mx^{m-1}y^3 - 2mx^{m-1}y^4 - anxy^{n-1} - any^n + anxy^n$$

To get the middle terms to simplify, we take $m = 2$ and $n = 4$, and $a = 1$:

$\dot{V} = -2x^2 + 4xy^3 - 4xy^4 - 4axy^3 - 4ay^4 + 4axy^4 = -2x^2 - 4y^4 < 0, \forall (x, y) \neq (0, 0)$

On the other hand

$V = x^2 + y^4 > 0, \forall (x, y) \neq (0, 0)$ and $V(0, 0) = 0$

In conclusion, the existence of the Lyapunov function implies global stability for the fixed point at the origin, hence no periodic solutions.