REVIEW
07/16/07

Solutions to Exam 2, Spring 2007

(1) (a) Consider a vector space V. An inner product on V is a function $\varphi : V\times V \longrightarrow \mathbb{R}$ with the following properties:

(i) $\varphi(cv+dw, u) = c\varphi(v,u) + d\varphi(w,u)$
$\varphi(v, cu+dw) = c\varphi(v,u) + d\varphi(v,w)$, for any $v, w, u \in V$ and $c, d \in \mathbb{R}$

(ii) $\varphi(v,w) = \varphi(w,v)$

(iii) $\varphi(v,v) \geq 0$, $(\forall)\, v\in V$ (equality iff $v=0$).

The function $\varphi : V\times V \longrightarrow \mathbb{R}$, $\varphi(v,w) = (v_1+v_2)(w_1+w_2)$ is not an inner product on $V=\mathbb{R}^2$, because condition (iii) (positivity) is not satisfied:

$\varphi(v,v) = (v_1+v_2)(v_1+v_2) = (v_1+v_2)^2 = 0 \Rightarrow v_2 = -v_1$

Hence $\varphi(v,v) = 0$ for any $v = \begin{pmatrix} a \\ -a \end{pmatrix}$, $a\in\mathbb{R}$, not only for $a=0$.

(b) A, B orthogonal matrices of size $(n\times n) \Longleftrightarrow A^T\cdot A = B^T\cdot B = I_n$

We calculate: $(AB)^T\cdot AB$ (note that AB is also a square $(n\times n)$ matrix):

$(AB)^T(AB) = (B^TA^T)(AB) = B^T(A^TA)B = B^TI_nB = B^TB = I_n$

Hence AB is orthogonal (by definition).

(2) (a) A is a (3×2) matrix, of rank $A = r$ ($r=1$ or $r=2$)

Then (FTLA):

$\dim(\text{colimg } A) = \dim(\text{img } A) = \text{rank } A = \text{rank } A^T = r$

$\dim(\ker A) = 2-r$, $\dim(\text{coker } A) = 3-r$

(b) The augmented matrix corresponding to $A=\begin{pmatrix} a_1 & a_2 \\ d_1 & d_2 \\ c_1 & c_2 \end{pmatrix}$, $b=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

is $M=\left(\begin{array}{cc|c} a_1 & a_2 & b_1 \\ d_1 & d_2 & b_2 \\ c_1 & c_2 & b_3 \end{array}\right)$. After Gaussian elimination we can turn the last row of A to zero. For compatibility of the system, the third value on the last row has to also be zero. For uniqueness of the solution, the two pivots have to be nonzero. Overall, unique solution to $Ax=b \iff \operatorname{rank}(A|b)=2$

(b') ~~Alternate solution~~: The Fundamental Theorem says that:

$$\dim(\ker A) = 2-r$$

If $Ax=b$ has unique solution, we have in particular that $\dim(\ker A)=0$ (because $\ker A=\{0\}$ in this case).

(c) $\dim(\operatorname{coker} A) = 2 \iff \operatorname{rank} A = 1 \iff \dim(\operatorname{rng} A) = 1$

We use the fact that $\operatorname{coker} A = (\operatorname{rng} A)^{\perp}$. Hence $\operatorname{rng} A$ is the orthogonal complement of $\operatorname{span}\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$. $w \in \operatorname{rng} A$ iff

$2w_1+w_2=0$ and $w_1+2w_2+w_3=0$

$\Rightarrow w_2=-2w_1$ and $w_3=-w_1-2w_2=-w_1+2\cdot 2w_1=3w_1 \Rightarrow$

$\Rightarrow \operatorname{rng} A = \left\{w_1\begin{pmatrix}1\\-2\\3\end{pmatrix}, w_1\in\mathbb{R}\right\} = \operatorname{span}\left\{\begin{pmatrix}1\\-2\\3\end{pmatrix}\right\}$

As expected, $\dim(\operatorname{rng} A)=1$.

In other words: $Ax=b$ has at least one solution iff $b\in\operatorname{rng} A$ iff $= \operatorname{span}\left\{\begin{pmatrix}1\\-2\\3\end{pmatrix}\right\}$ iff $b=a\cdot\begin{pmatrix}1\\-2\\3\end{pmatrix}$, for some $a\in\mathbb{R}$.

(d) See (c).

(3) $W = \operatorname{span}\{p_1(x) = 1,\ p_2(x) = x\} \subset \mathcal{P}^{(2)}$.

(a) The L^2 inner product on $\mathcal{C}^0[0,1]$ is a function $\varphi : (\mathcal{C}^0[0,1])^2 \longrightarrow \mathbb{R}$ given by: $\varphi(f,g) = \int_0^1 f(x)\,g(x)\,dx$, for any $f,g \in \mathcal{C}^0[0,1]$.

(b) If $W \subset V$ is a finite-dimensional subspace of V, we define the orthogonal complement of W to be $W^\perp = \{v \in V \mid \langle v,w\rangle = 0,\ \forall\, w \in W\} \subset V$ subspace.

In this case: $W^\perp = \{p \in \mathcal{P}^{(2)} \mid \langle p,p_1\rangle = \langle p,p_2\rangle = 0\}$ (Note that $\langle p, p_i\rangle = 0$ for $i = \overline{1,2}$ implies $\langle p, q\rangle = 0$, $\forall\, q \in W$, because $\{p_1,p_2\}$ is a basis of W.

$$\langle p, p_1\rangle = \int_0^1 p(x)\cdot p_1(x)\,dx = \int_0^1 p(x)\,dx = 0$$

$$\langle p, p_2\rangle = \int_0^1 p(x)\,p_2(x)\,dx = \int_0^1 x\,p(x)\,dx = 0$$

But $p(x) = a + bx + cx^2 \Rightarrow \int_0^1 p(x)\,dx = ax + \frac{bx^2}{2} + \frac{cx^3}{3}\Big|_0^1 =$

$$= a + \tfrac{1}{2}b + \tfrac{1}{3}c = 0$$

$$\int_0^1 x\,p(x)\,dx = \int_0^1 (ax + bx^2 + cx^3)\,dx = \frac{ax^2}{2} + \frac{bx^3}{3} + \frac{cx^4}{4}\Big|_0^1 = \frac{a}{2} + \frac{b}{3} + \frac{c}{4} = 0$$

We need to solve the system in a, b, c:

$$\begin{cases} 6a + 3b + 2c = 0 \\ 6a + 4b + 3c = 0 \end{cases} \Longrightarrow \begin{pmatrix} 6 & 3 & 2 \\ 6 & 4 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Augmented matrix: $\left(\begin{array}{ccc|c} 6 & 3 & 2 & 0 \\ 6 & 4 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 6 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$

one free variable: $c = \zeta \Rightarrow b = -\zeta \Rightarrow 6a + 3(-\zeta) + 2\zeta = 0 \Rightarrow$

$6a = \zeta \Rightarrow a = \zeta/6$

Hence $p(x) = \xi/6 - \xi x + \xi x^2 = \xi\left(\frac{1}{6} - x + x^2\right)$, $\xi \in \mathbb{R}$

$W^\perp = \{\xi(1/6 - x + x^2) / \xi \in \mathbb{R}\} = \text{span}\{\frac{1}{6} - x + x^2\}$

$\dim(W^\perp) = 1$ (Note that $\dim W = 2$ and $\dim W + \dim W^\perp = 3 = \dim \mathcal{P}^{(2)}$).

(4) (a) $\ker K = \{0\} \iff \text{rank } K = 2 \iff K$ nonsingular $\iff 1 \cdot d - 2 \cdot 2 = 0 \iff d = 4$

(b) A square matrix K is positive definite iff $x^T K x \geq 0$, $(\forall)\, x \in \mathbb{R}^n$ (where n is the size of the matrix), with equality iff $x = 0 \in \mathbb{R}^n$.

In our case, K positive definite iff:

$$q(x_1, x_2) = (x_1 \;\; x_2)\begin{pmatrix} 1 & 2 \\ 2 & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > 0, \text{ for any } (x_1, x_2) \neq (0,0).$$

$$q(x_1, x_2) = (x_1 \;\; x_2)\begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + d x_2 \end{pmatrix} = x_1^2 + 4x_1x_2 + d x_2^2 =$$

$$= (x_1 + 2x_2)^2 + (d-4)x_2^2$$

For $q(x_1, x_2) > 0$ for all $(x_1, x_2) \neq (0,0)$, we need $d - 4 > 0 \Rightarrow d > 4$. $d > 4$ it also sufficient.

Conclusion K positive definite iff $d > 4$.

(c) $\langle v, w \rangle = v^T K w$ is an inner product $\iff K$ positive definite $\iff d > 4$.

(d) Chose $d = 5 \Rightarrow K = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

$$\langle v, w \rangle_K = v^T K w = (v_1 \;\; v_2)\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

For $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow (1 \;\; 0)\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \Rightarrow (1 \;\; 0)\begin{pmatrix} w_1 + 2w_2 \\ 2w_1 + 5w_2 \end{pmatrix} = 0$

$$\Rightarrow w_1 + 2w_2 = 0 \Rightarrow w_1 = -2w_2 \Rightarrow w = \begin{pmatrix} -2w_2 \\ w_2 \end{pmatrix} = w_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Note that this is valid for all values of d that make $v^T K w$ an inner product. If we call $W = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$, we get that,

$W^\perp = \text{span}\left\{\begin{pmatrix}-2\\1\end{pmatrix}\right\}$. $(\dim W + \dim W^\perp = 2 = \dim \mathbb{R}^2)$.

(5) (a) $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$

If we attempted to solve the system, we would set up the augmented matrix:

$(K|b) = \left(\begin{array}{cc|c}1 & 2 & 2\\ 2 & 4 & 2\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}1 & 2 & 2\\ 0 & 0 & -2\end{array}\right) \Rightarrow$ no solution. (the system is incompatible)

We will be looking for the least squares solution, i.e. for the $x = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ such that $\|Kx - b\|$ is minimal.

$$q(x_1,x_2) = \left\|\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} - \begin{pmatrix}2\\2\end{pmatrix}\right\| = \left\|\begin{pmatrix}x_1+2x_2\\2x_1+4x_2\end{pmatrix} - \begin{pmatrix}2\\2\end{pmatrix}\right\| =$$

$$= (x_1+2x_2-2)^2 + (2x_1+4x_2-2)^2 =$$

$$= \cancel{5x_1^2 +} (x_1+2x_2)^2 - 4(x_1+2x_2) + 4 + 4(x_1+2x_2)^2 - 8(x_1+2x_2)+4$$

$$= 5(x_1+2x_2)^2 - 12(x_1+2x_2) + 8 = \left[(\sqrt{5}(x_1+2x_2))^2 - 2\cdot\sqrt{5}(x_1+2x_2)\cdot\frac{6}{\sqrt{5}} + \left(\frac{6}{\sqrt{5}}\right)^2\right] + 8 - \frac{36}{5} = \left[\sqrt{5}(x_1+2x_2) - \frac{6}{\sqrt{5}}\right]^2 + \frac{4}{5}$$

The minimum is obtained for $x_1 + 2x_2 = \frac{6}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}} = \frac{6}{5}$

In other words, there is a whole ~~subspace~~ that minimizes the norm:

$$W = \left\{\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2 \,/\, x_1 = \frac{6}{5} - 2x_2\right\} = \left\{\begin{pmatrix}\frac{6}{5}-2x_2\\x_2\end{pmatrix}, x_2 \in \mathbb{R}\right\} =$$

$$= \left\{\begin{pmatrix}6/5\\0\end{pmatrix} + x_2\begin{pmatrix}-2\\1\end{pmatrix} \,\middle|\, x_2 \in \mathbb{R}\right\}$$ (this is because K is only positive semidefinite, not positive definite) The error value is $\frac{4}{5}$.

(b) See above: $x \in \left\{ \begin{pmatrix} 6/5 \\ 0 \end{pmatrix} + \xi \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$

Any solution x to the least squares problem can be written as

$x = w + z$, where $w = \begin{pmatrix} 6/5 \\ 0 \end{pmatrix}$ and $z = \xi \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Clearly,