Review - July 25, 2007

Final exam - Spring, 2007 - Solutions

(1)(a) $\begin{cases} 2x - 6y + 4z = -4 \\ -x + 3y - 2z = c \end{cases} \Rightarrow A = \begin{pmatrix} 2 & -6 & 4 \\ -1 & 3 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} -4 \\ c \end{pmatrix}$

(b) If A is an arbitrary $(m \times n)$ matrix, then we define:

$\operatorname{rng} A = \{ b \in \mathbb{R}^m \mid \text{there exists } x \in \mathbb{R}^n \text{ such that } Ax = b \} \subset \mathbb{R}^m$.

$\operatorname{coker} A = \ker(A^T) = \{ x \in \mathbb{R}^m \,/\, A^T x = 0 \in \mathbb{R}^n \} \subset \mathbb{R}^m$

For our particular matrix A:

(I) $b \in \operatorname{rng} A \iff$ the system $Ax = b$ is compatible

$$(A \mid b) = \left(\begin{array}{ccc|c} 2 & -6 & 4 & b_1 \\ -1 & 3 & -2 & b_2 \end{array}\right) \xrightarrow{I \times (+\frac{1}{2}) + II} \left(\begin{array}{ccc|c} 2 & -6 & 4 & b_1 \\ 0 & 0 & 0 & b_2 + \frac{b_1}{2} \end{array}\right)$$

Hence $b \in \operatorname{rng} A \iff b_2 + \frac{b_1}{2} = 0 \iff b_1 = -2b_2$

$$\operatorname{rng} A = \left\{ \begin{pmatrix} -2b_2 \\ b_2 \end{pmatrix} \,/\, b_2 \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^2$$

(II) $x \in \operatorname{coker} A \iff A^T x = 0 \iff$

$$\begin{pmatrix} 2 & -1 \\ -6 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff 2x_1 = x_2$$

$$\operatorname{coker} A = \left\{ \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} \,|\, x_1 \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Check: $\dim(\operatorname{rng} A) = 1$, $\dim(\operatorname{coker} A) = 1$, $\dim(\operatorname{rng} A) + \dim(\operatorname{coker} A) = 2$

$$\left\langle \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle = 0 \Rightarrow \operatorname{rng} A \perp \operatorname{coker} A \rightarrow \operatorname{rng} A = (\operatorname{coker} A)^{\perp}$$

(c) The Fredholm compatibility conditions for $Ax = b$:

$b \in \operatorname{rng} A \iff \langle b, y\rangle = 0$, (∀) $y \in \operatorname{coker} A \iff \langle b, y_j\rangle = 0$, ∀ $y_j \in$ basis of coker A.

In our case: $\operatorname{coker} A = \operatorname{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$. Hence $b \in \operatorname{rng} A \iff b_1 + 2b_2 = 0$ (same conditions as at (a)).

$\begin{pmatrix}-4\\c\end{pmatrix} \in \operatorname{rng} A \iff -4 + 2c = 0 \Rightarrow \boxed{c = 2}$

(d) For $c = 2$, the system becomes: $2x_1 - 6x_2 + 4x_3 = -4 \iff x_1 - 3x_2 + 2x_3 = -2$

$\iff x_1 = 3x_2 - 2x_3 - 2$. The solution set is: $S = \left\{\begin{pmatrix}3x_2 - 2x_3 - 2\\x_2\\x_3\end{pmatrix}\right\} \Rightarrow$

$\left.\begin{array}{l} \ker A = \operatorname{span}\left\{\begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\} \\ \operatorname{corng} A = \operatorname{span}\left\{\begin{pmatrix}1\\-3\\2\end{pmatrix}\right\}\end{array}\right\} \Rightarrow$ any solution $x \in \mathbb{R}^3$ can be written as $x = w + z$, where $w \in \operatorname{corng} A$, $w = \begin{pmatrix}1\\-3\\2\end{pmatrix}$ and $z \in \ker A$.

(e) The solution with minimal Euclidean norm is $w = \begin{pmatrix}1\\-3\\2\end{pmatrix} \in \operatorname{corng} A$.

(2) (a) A set of vectors $\{v_1, \dots v_n\}$ is called a basis of the vector space V iff it is linearly independent (i) and it spans V (ii).

(i) (∀) $c_1, \dots c_n \in \mathbb{R}$ such that $c_1 v_1 + \dots + c_n v_n = 0 \Rightarrow c_1 = \dots = c_n = 0$

(ii) (∀) $w \in V \Rightarrow (\exists)\ a_1 \dots a_n \in \mathbb{R}$ such that $w = a_1 v_1 + \dots + a_n v_n$

(b) (i) $c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4 = \begin{pmatrix} c_1 + c_2 k + c_3 & -c_1 - 3c_2 + c_4 k \\ c_2 - c_3 k - c_4 & 2c_3 - 2c_4 \end{pmatrix} = 0 \Rightarrow$

$$\Rightarrow c_3 = c_4 \Rightarrow c_2 = c_4(k+1) \Rightarrow c_1 = c_4 k - 3c_4(k+1) = c_4(k-3k-3) = c_4(-2k-3)$$

$$c_1 = -kc_2 - c_3 = -k(k+1)c_4 - c_4 = -(k^2+k+1)c_4$$

$$\Rightarrow \begin{cases} \text{either } c_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0 \\ \text{or } k^2+k+1 = 2k+3 \Rightarrow (k^2-k-2) = 0 \Rightarrow (k-2)(k+1) = 0 \Rightarrow k=-1 \text{ or } k=2 \end{cases}$$

For $k=2 \Rightarrow c_3 = c_4,\ c_2 = 3c_4,\ c_1 = -7c_4$

For $k=-1 \Rightarrow c_3 = c_4,\ c_2 = 0,\ c_1 = -c_4$

<u>Conclusion</u>: For $k=-1$ and $k=2$, $(c_1\ c_2\ c_3\ c_4)^T$ does not necessarily have to be identically zero $\Rightarrow$ the matrices are linearly dependent.

For $k \neq -1$ and $k \neq 2$, A_1, A_2, A_3 and A_4 are linearly independent. $\Rightarrow$ basis of $M_{2\times 2}$.

(c) We want to find c_1, c_2, c_3, c_4 such that:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} c_1+c_2+c_3 & -c_1-3c_2+c_4 \\ c_2-c_3-c_4 & 2c_3-2c_4 \end{pmatrix} \Rightarrow \begin{cases} c_1+c_2+c_3 = 1 \\ -c_1-3c_2+c_4 = 0 \\ c_2-c_3-c_4 = 0 \\ 2c_3-2c_4 = 2 \end{cases} \Rightarrow$$

In matrix form:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ -1 & -3 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & -4 & 4 \end{array}\right) \Rightarrow c_4 = -1,\ c_3 = \left(-\frac{1}{2} - \frac{1}{2}c_4\right)2 = -c_4 - 1 = 0$$

$$c_2 = \frac{1}{2}(c_3 + c_4 - 1) = \frac{-2}{2} = -1$$

$$c_1 = 1 - c_2 - c_3 = 1 + 1 - 0 = 2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = 2A_1 - A_2 - A_4$$

(3) (a) An $(n \times n)$ matrix A is called positive definite if it is symmetric and if, additionally: $x^T A x \geq 0$, $\forall\, x \in \mathbb{R}^n$, (with equality only for $x = 0$).

(b) (i) Calculate the quadratic form associated with A and complete the square: $q(x) = x^T A x$.
Show that $q(x) \geq 0$, with equality when $x = 0$.

(ii) Diagonalize the matrix A, if possible (either LDL^T or $Q \Lambda Q^T$ factorizations) and show that the diagonal matrices have positive entries.

(iii) Calculate the minors of A and show that they are all positive.

(c) $$q(x,y,z) = X^T \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{K} X - 2 \underbrace{\begin{pmatrix} 1 \\ 0 \\ -3/2 \end{pmatrix}}_{f}{}^{T} X + \underbrace{2}_{c}$$

Calculate the solution of $Kx = f$: $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -3/2 \end{array}\right) \longrightarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & 1 & -3/2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1/2 & -1 \end{array}\right) \Rightarrow z = -2,\ y = \frac{1}{2}(-1-z) = \frac{1}{2}$$

$$x = 1 - y = 1 - \frac{1}{2} = \frac{1}{2}$$

$$X^* = \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -2 \end{pmatrix}$$

(d) X^* is a global minimizer because $K = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1/2 \end{pmatrix} = U$ is positive definite (as its upper triangular form U shows, by having only strictly positive pivots).

(4)(a) $w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$. Calculate: $\begin{cases} \langle w_2, v_1 \rangle = 1 \\ \|v_1\|^{\cancel{2}} = \sqrt{2} \end{cases}$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \\ -1/2 \end{pmatrix}$$

(b) $w_1 = r_{11} u_1$, where $r_{11} = \|w_1\| = \sqrt{2}$ and so $u_1 = \dfrac{w_1}{\|w_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

$w_2 = r_{12} u_1 + r_{22} u_2$, where $r_{12} = \langle w_2, u_1 \rangle = \dfrac{1}{\sqrt{2}}$ and $r_{22} = \sqrt{\|w_2\|^2 - r_{12}^2} = \sqrt{2 - \frac{1}{2}} = \sqrt{\frac{3}{2}}$

$$u_2 = \frac{w_2 - r_{12} u_1}{r_{22}} = \sqrt{\frac{2}{3}} \cdot \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \right] = \sqrt{\frac{2}{3}} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ -1/2 \end{pmatrix} \right] =$$

$$= \sqrt{\frac{2}{3}} \begin{pmatrix} -1/2 \\ 1 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \sqrt{2/3} \\ 0 \\ -1/\sqrt{6} \end{pmatrix}$$

In conclusion, the QR factorization for B is:

$$Q = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix}$$

(5)(a) TRUE. Suppose v is an eigenvector of A, with eigenvalue $\lambda \neq 0$ (since A is invertible, all its eigenvalues are nonzero). Then:

$$Av = \lambda v \Rightarrow A^{-1}(Av) = A^{-1}\cdot(\lambda v) \Rightarrow (A^{-1}A)v = \lambda(A^{-1}v) \Rightarrow v = \lambda(A^{-1}v)$$

$$\Rightarrow A^{-1}v = \frac{1}{\lambda}v \rightarrow v \text{ is an eigenvector of } A^{-1} \text{ with eigenvalue } \frac{1}{\lambda}.$$

(b) $A^4 = I \Rightarrow (A^2 - I)(A^2 - I) = 0 \Rightarrow (A+I)(A-I)(A+iI)(A-iI) = 0 \Rightarrow$

$\Rightarrow \det[(A+I)(A-I)(A+iI)(A-iI)] = 0$ — Moreover:

Suppose λ is an eigenvalue of $A \Rightarrow \det(A - \lambda I) = 0 \Rightarrow \det[(A-\lambda I)(A+\lambda I)] = 0$

$\Rightarrow \det(A^2 - \lambda^2 I) = 0 \Rightarrow \det[(A^2 - \lambda^2 I)(A^2 + \lambda^2 I)] = 0 \Rightarrow \det(A^4 - \lambda^4 I) = 0 \Rightarrow$

$\Rightarrow \det(I - \lambda^4 I) = 0 \Rightarrow \det(1 - \lambda^4)I = 0 \Rightarrow 1 - \lambda^4 = 0 \Rightarrow \lambda^4 = 1 \Rightarrow \lambda = \pm 1$ or $\pm i$.

$\Rightarrow$ TRUE

(c) FALSE. Counterexample: $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -1 \\ 4 & 4 \end{pmatrix} \Rightarrow A + B = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}$

$\lambda = 3$ is an eigenvalue for both A and B:

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad B\begin{pmatrix} 1 \\ -4 \end{pmatrix} = 3\cdot\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

However; for $A+B$: $(2-\lambda)(6-\lambda) + 4 = \lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda \neq 3$.

If we require that the common eigenvalue must have the same eigenvector for A and B, then the statement is true:

$$Av = \lambda v,\ Bv = \lambda v \Rightarrow (A+B)v = \lambda v.$$

(d) FALSE: One of the eigenvalues may still be zero even when C is diagonalizable, making C singular. E.g.:

$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is already diagonal, but $\lambda_3 = 0 \Rightarrow C$ singular

(6)(a) corange $A = \text{span}\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\} \Rightarrow \ker A = \text{span}\left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\}$

$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$ with $\ker A = \text{span}\left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\} \Rightarrow a_j - 2b_j = 0 \Rightarrow A = \begin{pmatrix} 2b_1 & b_1 \\ 2b_2 & b_2 \\ 2b_3 & b_3 \end{pmatrix}$

But $\text{rng}\, A = \text{span}\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\} \Rightarrow \frac{b_2}{b_1} = 2$ and $\frac{b_3}{b_1} = -1 \Rightarrow A = b_1 \cdot \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{pmatrix}$

So choose $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{pmatrix}$ and check that the two subspaces are as required.

(b) (By FTLA, $\ker A \oplus \text{corng}\, A = \mathbb{R}^3$ (orthogonal complements) $\Rightarrow$
$\dim(\ker A) + \dim(\text{corng}\, A) = 3$. One of the two would have to be 2-dim)

$\begin{pmatrix}1\\1\\1\end{pmatrix} \in \text{corng}\, A \Rightarrow (\exists)\, x \in \mathbb{R}^3$ such that $A^T x = \begin{pmatrix}1\\1\\1\end{pmatrix} \Leftrightarrow x^T A = (1 \ 1 \ 1)$

But $\begin{pmatrix}1\\-1\\2\end{pmatrix} \in \ker A \Rightarrow A\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow x^T A \begin{pmatrix}1\\-1\\2\end{pmatrix} = 0$

On the other hand: $x^T A \begin{pmatrix}1\\-1\\2\end{pmatrix} = (1 \ 1 \ 1)\begin{pmatrix}1\\-1\\2\end{pmatrix} = 2$ } contradiction

there is no such matrix.

(c) there is no such matrix. A real, symmetric $\Rightarrow$ A has real eigenvalues.

(d) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$

$A\begin{pmatrix}x\\y\end{pmatrix} = 2\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow \begin{cases} 2x + y = 2x \\ 2y = 2y \end{cases} \Rightarrow y = 0 \Rightarrow W_\lambda = \left\{\begin{pmatrix}x\\0\end{pmatrix} \middle| x \in \mathbb{R}\right\} = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}.$