APPM 3310: Matrix Methods — Exam #2 — April 13, 2007

On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. Show all work in your bluebook. Textbooks, class notes and calculators are not permitted, although you are allowed to use an index card reminder sheet.

Please sign your bluebook under the Honor Code to indicate that you have neither given nor received unauthorized assistance on this exam.

1. (15 points) A couple of short-answer questions to get you started:
   (a) Give the definition of an inner product. Does $(v_1 + v_2)(w_1 + w_2)$ define an inner product $< \mathbf{v}, \mathbf{w} >$ on $\mathbb{R}^2$? Explain.
   (b) True or False: If A and B are both orthogonal matrices of the same size, then so is AB. Explain. (Your answer should include the definition of an orthogonal matrix.)

2. (25 points) For this problem, assume A is a 3×2 matrix, and consider the system $A\mathbf{x} = \mathbf{b}$.
   (a) State the Fundamental Theorem of Linear Algebra.
   (b) Under what conditions on A and $\mathbf{b}$ will the system have exactly one solution? Briefly (no more than 1-2 phrases or sentences) explain how these conditions are related to the Fundamental Theorem.
   (c) Now, suppose $\operatorname{coker}(A) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$. What conditions must $\mathbf{b}$ satisfy for at least one solution to exist?
   (d) Assuming $\operatorname{coker}(A)$ is given in part (c), what is $\operatorname{rank}(A)$? What is $\dim(\operatorname{corange}(A))$? What is $\operatorname{rng}(A)$?

3. (25 points) Let $W = \operatorname{span}\{p_1(x) = 1, p_2(x) = x\}$ be a subspace of $P^{(2)}$.
   (a) Give the definition of the $\mathbf{L_2}$**-inner product** on the interval $[0, 1]$.
   (b) Define $W^\perp$, the orthogonal complement to W.
   (c) Find a basis for and the dimension of $W^\perp$, using the inner product you defined in part (a).

4. (25 points) Let $K = \begin{pmatrix} 1 & 2 \\ 2 & d \end{pmatrix}$.
   (a) Find all values of d for which $\ker(K) = \{\mathbf{0}\}$.
   (b) Give the definition for a matrix to be positive definite. Now, find all values of d for which K is positive definite.
   (c) Find all values of d for which $< \mathbf{v}, \mathbf{w} >= \mathbf{v}^T K \mathbf{w}$ is an inner product.
   (d) Find a vector orthogonal to $\mathbf{v} = (1, 0)^T$ using the inner product in (c) with one of your acceptable values of d. (Choose any d if you didn't get (c).)

5. (20 points) Use the same matrix K as in the previous problem, with $d = 4$.
   (a) Find the least squares solution to $K\mathbf{x} = (2, 2)^T$.
   (b) Write your solution in the form $\mathbf{x} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w} \in \operatorname{corange}(K)$ and $\mathbf{z} \in \ker(K)$.