APPM 3310: Matrix Methods — Final Exam — May 8, 2007

On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. Show all work in your bluebook. Textbooks, class notes and calculators are not permitted, although you are allowed to use one page of notes as a reminder sheet. If you find that the arithmetic for this exam seems complicated, go back and check your work.

Please sign your bluebook under the Honor Code to indicate that you have neither given nor received unauthorized assistance on this exam.

1. (25 points) Consider the set of equations $2x - 6y + 4z = -4$, $\quad -x + 3y - 2z = c$, where c is some real number.
   (a) Write the system in matrix form $A\mathbf{x} = \mathbf{b}$.
   (b) Define the fundamental subspaces **range** and **cokernel** for an arbitrary $m \times n$ matrix. Find these spaces for A.
   (c) Define the **Fredholm compatibility conditions** (Fredholm alternative) for a general system $A\mathbf{x} = \mathbf{b}$. Find a value of c that satisfies these conditions for the system in (a).
   (d) For this c value, find the general solution to (a). Write the solution as $\mathbf{x} = \mathbf{w} + \mathbf{z}$, where $\mathbf{w} \in \text{corange}(A)$ and $\mathbf{z} \in \ker(A)$.
   (e) Of the solutions in (d), which has the smallest Euclidean norm?

2. (25 points) For this problem, let $A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} k & -3 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 \\ -k & 2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & k \\ -1 & -2 \end{bmatrix}$ be four elements in $M_{2\times 2}$.
   (a) Give the **complete** definition for a set of vectors to be a basis of a vector space. (This definition should be at least two or three sentences long, it should not be just one or two words.)
   (b) Find all values of the scalar k for which the set $\{A_1, A_2, A_3, A_4\}$ forms a basis of $M_{2\times 2}$.
   (c) Now, let $k = 1$. Write the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ as a linear combination of $\{A_1, A_2, A_3, A_4\}$, if possible. If it is not possible, explain why.

3. (25 points) Consider the quadratic function

$$q(x, y, z) = x^2 + 2xy + 3y^2 + 2yz + z^2 - 2x + 3z + 2.$$

   (a) Give the definition of a positive definite matrix.
   (b) Describe (in words) at least two different ways for determining whether a matrix is positive definite or not.
   (c) Find (x, y, z) that minimizes the quadratic function $q(x, y, z)$.
   (d) Briefly explain how you know that your answer in part (c) is really the global minimum.

4. (15 points) Define $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$.

   (a) Apply the Gram-Schmidt algorithm to the columns of matrix B.

   (b) Find the QR factorization of matrix B, where Q is an orthonormal matrix and R is upper triangular.

5. (26 points) True or False. If True, explain why the statement is true. If False, give an example to show why it is false.

   (a) Assume matrix A is $n \times n$ and A^{-1} exists. Then, the eigenvectors of A and A^{-1} are the same.

   (b) If A is a matrix such that $A^4 = I$, then the only possible eigenvalues of A are 1, -1, i, and $-i$.

   (c) If λ is an eigenvalue of both A and B, then it is an eigenvalue of $A + B$.

   (d) If a matrix C is diagonalizable, then it is also nonsingular.

6. (24 points) For each property given below, write down a matrix A with that property or explain why no such matrix exists.

   (a) $\text{rng}(A) = \text{span}\{(1, 2, -1)^T\}$, and $\text{corange}(A) = \text{span}\{(2, 1)^T\}$.

   (b) The vector $(1, -1, 2)^T$ is in the kernel of the A, $(1, 1, 1)^T$ is in the corange of A, and $\det A = 1$.

   (c) A is real and symmetric with eigenvalues $1 + i$ and $1 - i$.

   (d) A has an eigenvalue 2 with algebraic multiplicity two, but geometric multiplicty 1 (i.e. only one linearly independent eigenvector).

7. (20 points) Suppose A is a 3×2 matrix for which
   (i) $A^T A$ has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ and corresponding eigenvectors $\mathbf{v}_1 = (1, 1)^T$ and $\mathbf{v}_2 = (1, -1)^T$, and
   (ii) AA^T has nonzero eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ and corresponding eigenvectors $(2, 1, 1)^T$ and $(0, 1, -1)^T$.

   (a) What is the eigenvector of AA^T that corresponds to the zero eigenvalue?

   (b) What is the singular value decomposition of A?

   (c) Using the SVD, write A as the sum of rank 1 matrices.

Extra Credit (up to 5 points). What is the most important thing (or useful to your major) you've learned this semester in this class?