Prof. Vanja Dukic, Department of Applied Mathematics
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1 Binomial Random Variable: Mean and Variance

We say that a random variable X is binomially distributed with probability p, and write $X \sim \text{Bin}(n,p)$, if its probability mass function (pmf) is given as follows:

$$P(X=x) = \binom{n}{x} p^x (1-p)^{(n-x)}$$

Here, X counts the total number of successes (1's) in n independent and identically distributed Bernoulli experiments, where the probability of success (or 1) at each experiment is p. Consequently, we can write $X = \sum_{i=1}^{n} X_i$ where each $X_i \sim \text{Bernoulli}(p)$.

The mean of each Bernoulli component can be computed as follows:

$$E(X_i) = 0(1-p) + 1p = 0 + p = p$$

and from here,

$$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np$$

Similarly, the variance of each Bernoulli component can be computed as follows:

$$V(X_i) = (0-p)^2(1-p) + (1-p)^2 p = p^2(1-p) + (1-p)^2 p = p(1-p)(p+1-p) = 1(1-p)$$

and from here,

$$V(X) = \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} p(1-p) = np(1-p)$$

We can simply sum the variances of the Bernoulli variables because they are all independent.

Alternatively, we can find the mean using some simple algebra and the Binomial Theorem:

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{(n-x)} =$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)} =$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{(x-1)} (1-p)^{(n-x)}$$

$$= np \sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-1-y)!} p^y (1-p)^{(n-1-y)} = np$$

Similarly, we can find the mean using some simple algebra and the Binomial Theorem:

$$V(X) = E(X^2) - E(X)^2 = E(X(X-1)) + E(X) - E(X)^2 =$$

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$$= \sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{(n-x)} + np - n^2p^2 =$$

$$= \sum_{x=2}^{n} x(x-1)\frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)} + np - n^2p^2 =$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{(x-2)} (1-p)^{(n-x)} + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{y=0}^{n-2} \frac{(n-2)!}{y!(n-2-y)!} p^{y} (1-p)^{(n-2-y)} =$$

$$= n(n-1)p^2 + np - n^2p^2 = np(1-p)$$

2 Negative Binomial Random Variable: Mean and Variance

We say that a random variable X is negative-binomial with probability p and parameter r, and write $X \sim \text{NegBin}(r, p)$, if its probability mass function (pmf) is given as follows:

$$P(X = x) = \binom{x+r-1}{x} p^r (1-p)^x$$

Here, X counts the total number of failures (0's) before observing r-th success in a sequence of independent and identically distributed Bernoulli experiments where the probability of success (or 1) at each experiment is p.

Alternatively, we can write $X = \sum_{i=1}^{r} X_i$ where each independent $X_i \sim \text{Geom}(p)$ counts the number of failures between $(i-1)$st and ith success.

The mean of a Negative Binomial random variable can be computed as follows:

$$E(X) = \sum_{x=0}^{\infty} x\binom{r+x-1}{x} p^r (1-p)^x =$$

$$\sum_{x=1}^{\infty} \frac{(r+x-1)!}{(x-1)!(r-1)!} p^r (1-p)^x =$$

$$\frac{r(1-p)}{p} \sum_{x=1}^{\infty} \frac{(r+x-1)!}{(x-1)!r!} p^{(r+1)} (1-p)^{(x-1)} =$$

$$\frac{r(1-p)}{p} \sum_{y=0}^{\infty} \binom{r+1+y-1}{y} p^{(r+1)} (1-p)^{y} =$$

$$= \frac{r(1-p)}{p}$$

The variance is computed in a similar way.

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3 Poisson Random Variable: Mean and Variance

A random variable X is said to be Poisson distributed with mean λ if its pmf is given as

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

for any $x = 0, 1, ...\infty$.
The mean of a Poisson random variable can be found via:

$$E(X) = \sum_{x=0}^{\infty} xe^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} xe^{-\lambda}\frac{\lambda^x}{x!} =$$

$$= \sum_{x=1}^{\infty} e^{-\lambda}\frac{\lambda^x}{(x-1)!} = \lambda\sum_{y=0}^{\infty} e^{-\lambda}\frac{\lambda^y}{y!} = \lambda$$

And similarly, the variance can be found via:

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1)e^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=2}^{\infty} x(x-1)e^{-\lambda}\frac{\lambda^x}{x!} =$$

$$= \sum_{x=2}^{\infty} e^{-\lambda}\frac{\lambda^x}{(x-2)!} = \lambda^2\sum_{y=0}^{\infty} e^{-\lambda}\frac{\lambda^y}{y!} = \lambda^2$$

From here,

$$V(X) = E(X(X-1)) + E(X) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

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