

# APPLIED ANALYSIS PRELIMINARY EXAMINATION

August 19, 1994

Instructions: You have three hours to complete this exam. Please start each problem on a new page.

## Do one of the following two:

1. Two functions  $f$  and  $g$  are defined as follows:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad f(x) = (x_1 \cos(x_2), \quad x_1^2 + x_2^2, \quad x_1 x_2)$$

$$g : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad g(x) = (x_1 x_2 \cos(x_3), \quad x_1 x_2 x_3)$$

Compute the Jacobian matrix  $D(g \circ f)(0, \pi)$  using the chain rule. (No credit for forming the composition and then computing the Jacobian.)

2. Given  $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}^1$  continuously differentiable functions

$$\text{Let } X = \{x \in \mathbb{R}^3 : g(x) = 0\}.$$

Suppose that  $x_0$  is a relative minimum of the function  $f$  restricted to  $X$ . Show that  $\nabla f(x_0)$  is a scalar multiple of  $\nabla g(x_0)$ , provided  $\nabla g(x_0) \neq 0$ .

## Do four of the following five:

3. Suppose  $f_n, f \in L^2(1, +\infty)$  and  $f_n \rightarrow f$  pointwise almost everywhere. Find the relationship between the following integrals and prove your claims.

$$(a) \int_1^{+\infty} \frac{f^2}{1+f^2}, \quad \underline{\lim} \int_1^{\infty} \frac{f_n^2}{1+f_n^2}, \quad \overline{\lim} \int_1^{\infty} \frac{f_n^2}{1+f_n^2}$$

$$(b) \int_1^{+\infty} \frac{f^2}{1+x^2 f^2}, \quad \underline{\lim} \int_1^{\infty} \frac{f_n^2}{1+x^2 f_n^2}, \quad \overline{\lim} \int_1^{\infty} \frac{f_n^2}{1+x^2 f_n^2} .$$

4. Let  $H = L^2[0, 2\pi]$ . An operator  $A : H \rightarrow H$  is defined by

$$(Af)(x) = \int_0^{2\pi} (\sin x \cos y + \cos y) f(y) dy.$$

Answer the following questions with proofs:

(a) Is  $A$  self-adjoint?

(b) Is  $A$  compact?

5. Given functions  $f_n, f \in H = L^2[0, 1]$ ,  $f_n \rightharpoonup f$

Show that  $f_n \rightarrow f$  if and only if  $\overline{\lim}_{n \rightarrow \infty} \int |f_n|^2 = \int f^2$ .

6. Prove there exists a unique continuous solution  $u$  to the ODE:

$$u'(t) = \frac{\sin(t u(t))}{2t} \quad \text{for } 0 < t \leq 1 \text{ which satisfies the initial condition } u(0) = 1.$$

7. Let  $S$  be the unit sphere in  $L^2[0, 2\pi]$ .

Define a function in  $G : S \rightarrow \mathbb{R}^1$  by  $G(f) = \int_0^{2\pi} f(x) \sin(x) dx$ .

Calculate the  $\sup \{G(f) : f \in S\}$  and calculate the  $\inf \{G(f) : f \in S\}$ . Justify your calculations!