APPLIED ANALYSIS PRELIMINARY EXAMINATION

August 23, 1995

Instructions:

You have three hours to complete this exam.

Work five of the six problems.

Please start each problem on a new page.

Write your name on your exam.

1. State Green's theorem on a domain $\Omega \subset R^2$. Le u and v be C^2 real valued functions which vanish on the boundary of Ω. Prove that

   (a) $\int_\Omega v\Delta u = \int_\Omega u\Delta v$

   (b) $\int_\Omega u\Delta u = 0$ implies that u vanishes on Ω.

2. Let $\{a_n\}$ be a sequence of non-negative numbers. Show that $b_n = \sqrt{a_1 + \sqrt{a_2 + \ldots + \sqrt{a_n}}}$ is convergent if and only if $\left\{a_n^{\frac{1}{2^n}}\right\}_{n=1}^{\infty}$ is bounded.

   (Hint: b_n is monotone non-decreasing.)

3. Prove the implicit function theorem:

   If $f : R^2 \to R^1$ is C^1 with

$$f(0,0) = 0, \qquad \frac{\partial f(x,y)}{\partial y}\Big|_{x=y=0} = 1, \qquad \frac{\partial f(x,y)}{\partial x}\Big|_{x=y=0} = 0$$

   then, there exists $\delta > 0$, $\phi : (-\delta, \delta) \to R$ such that $f(x, \phi(x)) = 0$, with $\phi(0) = 0$, $\phi \in C^1$.

4. f_n are measurable functions on $[0, +\infty)$, $f_n \to 0$ a.e. on $[0, +\infty)$, and f is a function in $L^2[0, +\infty)$. Show that

$$\int_1^{+\infty} f_n(x) f(x) dx \to 0 \text{ as } n \to +\infty$$

   provided that $|f_n(x)| \le x^{-\alpha}$ for some $\alpha > \frac{1}{2}$.

5. Let T be a compact ooperator from a Banach space X to a Hilbert Space H. Show that there exists $T_n \in B(X, H)$, such that

(a) $\dim R(T_n) < +\infty$

(b) $||T_n|| \le ||T||$

(c) $||T_n - T|| \to 0$ as $n \to +\infty$.

6. Let $L : H \to H$ be a self adjoint linear operator on a Hilbert space H. Suppose that L is positive definite in the sense that $\langle Lu, u\rangle > 0$ for all non-zero vectors u. Define an "energy" norm on H by setting $||u|| = \langle Lu, u\rangle$. Let S be the subspace of H spanned by the linearly independent vectors $v_1, \ldots, v_N$. Let $f \in H$ and suppose that $Lx = f$. Show that among all vectors in S, the vector $U \in S$ which is closest to x in the energy norm must satisfy the system of equations

$$\langle LU, v_j\rangle = \langle f, v_j\rangle \text{ for } 1 \le j \le N.$$

(Thus U is the best approximate solution in S of the equation $Lx = f$.)