APPLIED ANALYSIS PRELIMINARY EXAMINATION

January 10, 1996

Instructions:

You have three hours to complete this exam. Work five of the six problems. Please start each problem on a new page. Write your name on your exam.

1. Prove that the sequence of functions $f_n : [0, 2\pi] \to R^1, \quad f_n(x) = \cos(nx)$, has no subsequence converging uniformly.

2. Find the maximum value of the function $f(x) = x_1^2 x_2^2 x_3^2$ on the unit sphere $\{x_1^2 + x_2^2 + x_3^2 = 1\}$. Justify your answer.

3. Prove the implicit function theorem:

   If $f : R^3 \to R^1$ is C^1 with

$$f(0,0,0) = 1, \qquad \frac{\partial f(x,y,z)}{\partial z} \Big|_{x=y=z=0} = -1$$

   then, there exists $\delta > 0$, $\phi : B_\delta \subset R^2 \to R$ such that $f(x, y, \phi(x, y)) = 1$, with $\phi(0, 0) = 0$, $\phi \in C^1$.

4. f_n and g_n are two sequences of integrable functions $[1, \infty) \longrightarrow R$, s.t. $f_n \to \frac{1}{x^2}$ a.e., $g_n \to \frac{1}{x^2}$ in measure (both are w.r.t. the Lebesgue measure on $[1, \infty)$), and $g_n \geq 0$. Give a short proof or a counter-example to the following statements:

   (a) $\liminf \int_{[1,\infty)} f_n \geq 1$

   (b) $\liminf \int_{[1,\infty)} g_n \geq 1$

   (c) There is a subsequence $\{f_{n_k} g_{n_k}\}$ of $\{f_n g_n\}$ s.t. $f_{n_k} g_{n_k} \to 1$ in measure in $[1, \infty)$.

5. (a) Find the spectral family of the operator $A : H = L^2(-1, 1) \to H$ defined by $(Af)(t) = \frac{1}{t+1} f(t)$.

   (b) ϕ is a function given by $\phi(s) = s^2$ when $s \leq -1$ and $\phi(s) = 1$ when $s > 0$. What is $\phi(A)$?

6. Use the contraction mapping theorem to prove that there exists a unique C^1 solution u to the ODE:

   $u'(t) = \sin(t\, u(t)) + t\cos(t^2) \quad$ for $\quad -\delta < t \leq \delta$ for some $\delta > 0$ with the initial condition $u(0) = 1$.