APPLIED ANALYSIS PRELIMINARY EXAMINATION
Aug. 20, 1997

Instructions:

You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You **MUST** prove your conclusions or show a counter-example for all problems. Write your name on your exam.

1. (a) Compute the line integral
$$\oint_C \cos(y)\,dx + (e^y + cos(x))\,dy$$
in the counter-clockwise direction. Here C is the unit circle $x^2 + y^2 = 1$.
   (b) Compute $\lim_{n\to\infty} \Sigma_{k=1}^{k=n} \dfrac{k^3}{n^4}$
   (c) Find $Max(2x + 2y + z)$ and $Min(2x + y + 2z)$ on the surface $x^2 + y^2 + z^2 = 1$.

2. Let $f \in L^1([0,\infty))$ and define $g(t) = \int_0^\infty e^{-xt} f(x)dx$. Prove:
   (a) $g(t)$ is differentiable
   (b) $g'(t) = \int_0^\infty -xe^{-xt} f(x)dx$.

3. $T : X = L^2([0,1]) \to X$ is defined by $(Tf)(x) = f(x) + \displaystyle\int_0^x (\sin t^2) f(t)dt$.
   (a) Is T a compact operator?
   (b) Find all eigenvalues of T.
   (c) Does 1 belong to the continuous spectrum of T?

4. Prove the existence and uniqueness of a C^2 solution to the initial value problem
$$\begin{cases} u''(t) + \frac{3}{1+t}u' = -e^u \\ u(0) = 1 \\ u'(0) = 0. \end{cases}$$
for $t \in [0, \delta]$, for some $\delta > 0$.

5. Let μ denote the Lebesgue measure on R^1, let $\{f_n\}_{n=1}^\infty$ be a sequence of integrable functions $R^1 \longrightarrow R$. All the following problems are on R^1 w.r.t the Lebesgue measure.
   (a) Define what is meant by $f_n \to f$ as $n \to \infty$ in measure.
   (b) Determine if true or false of the following statements. If false, give counter-examples. You don't have to provide proofs for the true statements.
   (i) If $f_n \to f$ a.e, then $f_n \to f$ in measure.
   (ii)If $f_n \to f$ in measure, then $f_n \to f$ a.e.
   (iii) If$f_n \to f$ in measure then, there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.u. as $k \to \infty$.
   (iv) If$f_n \to f$ in measure, then $\int_{R^1} |f_n(x) - f(x)| d\mu \to 0$.
   (v) If$f_n \to f$ in measure and $f_n \geq 0$, then $\int_{R^1} \underline{\lim}_{n\to\infty} f_n \geq \int f$.